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An Index for $2D$ field theories with large $\mathcal{N} = 4$ superconformal symmetry

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Abstract

We consider families of theories with large $\mathcal{N} = 4$ superconformal symmetry. We define an index generalizing the elliptic genus of theories with $\mathcal{N} = 2$ symmetry. In contrast to the $\mathcal{N} = 2$ case, the new index constrains part of the non-BPS spectrum. Motivated by aspects of the AdS/CFT correspondence we study the index in the examples of symmetric product theories. We give a physical interpretation of the Hecke operators which appear in the expressions for partition functions of such theories. Finally, we compute the index for a nontrivial example of a symmetric product theory.

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1. Introduction and Summary

The elliptic genus of an $\mathcal{N}=2$ superconformal field theory is a powerful tool in analyzing the field content of subtle conformal field theories (such as CY sigma models) that depend on parameters [1,2,3,4,5,6,7,8,9,10,11]. The reason it is useful is that it is invariant under deformations of parameters that preserve the $\mathcal{N}=2$ supersymmetry. Usually, one can deform parameters to a region where the genus can be evaluated explicitly. One then obtains some nontrivial information about the CFT for all values of parameters. In this paper we will describe an analog of the elliptic genus for theories with a larger superconformal symmetry, namely, the maximal, or large $\mathcal{N}=4$ superconformal symmetry $\mathcal{A}_\gamma$ discovered in [12]. Some background on the large superconformal algebra can be found in [13,14,15,16,17,18]. We use the conventions described in [18].

In conformal field theories with $\mathcal{A}_\gamma$ symmetry one can expect that the larger amount of symmetry leads to more control of the spectrum. Here we show that this is indeed the case. In equations (3.3) and (3.4) below we define an index for theories with $\mathcal{A}_\gamma$ symmetry which is invariant under deformations preserving $\mathcal{A}_\gamma$ symmetry. The index satisfies some novel properties reflecting the greater control on the spectrum. For example, it is not holomorphic, and does not just count BPS states.

A natural operation on conformal field theories is the symmetric product orbifold. This operation has proven to be of great importance in Matrix string theory and in the AdS/CFT correspondence. It is therefore natural to study the $\mathcal{A}_\gamma$ index for such theories. If $\mathcal{C}_0$ is a conformal field theory with $\mathcal{A}_\gamma$ symmetry then it turns out that there is a remarkably simple formula for the index of $\mathrm{Sym}^N(\mathcal{C}_0)$. This is equation (4.6) below, which expresses the index in terms of a Hecke transform of the partition function of $\mathcal{C}_0$. The occurance of Hecke operators in formulae for partition functions of symmetric product theories has been noted previously [19]. In section 4.2 below we give a simple physical interpretation to some aspects of the Hecke algebra in terms of "short string" and "long string" operators.

In section 5 we describe the computation of the index in a nontrivial example, $\mathrm{Sym}^N(\mathcal{S})$, where $\mathcal{S}$ is the simplest theory with $\mathcal{A}_\gamma$ symmetry. We also describe some partial results for some other theories with $\mathcal{A}_\gamma$ symmetry in section 6. Several definitions and conventions on theta functions are relegated to the appendices.

This work was motivated by our search for a holographic dual to type II string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$. Applications to that problem appear in [18]. However, since the

results on the index might be of use in other applications of superconformal field theory we have written a separate paper. For other applications of elliptic genera to the AdS/CFT correspondence see [20,21,22,23,24].

2. Prelimimaries

In this section we briefly review some aspects of the $\mathcal{A}_\gamma$ algebra and establish our notation.

2.1. The superconformal algebra $\mathcal{A}_\gamma$

Apart from the usual Virasoro algebra, the large $\mathcal{N}=4$ superconformal algebra $\mathcal{A}_\gamma$ contains two copies of the affine $\widehat{SU(2)}$ Lie algebras, at the levels k^+ and k^-, respectively. We also define $k \equiv k^+ + k^-$. The relation between $k^\pm$ and the parameter γ is

$$\gamma = \frac{k^-}{k^+ + k^-} \ . \tag{2.1}$$

Unitarity implies that the Virasoro central charge is:

$$c = \frac{6k^+k^-}{k^+ + k^-} \ . \tag{2.2}$$

The superconformal algebra $\mathcal{A}_\gamma$ is generated by six affine $\widehat{SU(2)}$ generators $A^{\pm,i}(z)$, four dimension 3/2 supersymmetry generators $G^a(z)$, four dimension 1/2 fields $Q^a(z)$, a dimension 1 field $U(z)$, and the Virasoro current $T(z)$. The OPEs with the Virasoro

generators, T_m, have the usual form. The remaining OPEs are [12,25]:

$$\begin{aligned}
G^a(z)G^b(w) &= \frac{2c}{3}\frac{\delta^{ab}}{(z-w)^3} - \frac{8\gamma\alpha^{+,i}_{ab}A^{+,i}(w) + 8(1-\gamma)\alpha^{-,i}_{ab}A^{-,i}(w)}{(z-w)^2} - \\
&\quad - \frac{4\gamma\alpha^{+,i}_{ab}\partial A^{+,i}(w) + 4(1-\gamma)\alpha^{-,i}_{ab}\partial A^{-,i}(w)}{z-w} + \frac{2\delta^{ab}L(w)}{z-w} + \ldots , \\
A^{\pm,i}(z)A^{\pm,j}(w) &= -\frac{k^{\pm}\delta^{ij}}{2(z-w)^2} + \frac{\epsilon^{ijk}A^{\pm,k}(w)}{z-w} + \ldots , \\
Q^a(z)Q^b(w) &= -\frac{(k^+ + k^-)\delta^{ab}}{2(z-w)} + \ldots , \\
U(z)U(w) &= -\frac{k^+ + k^-}{2(z-w)^2} + \ldots , \\
A^{\pm,i}(z)G^a(w) &= \mp\frac{2k^{\pm}\alpha^{\pm,i}_{ab}Q^b(w)}{(k^+ + k^-)(z-w)^2} + \frac{\alpha^{\pm,i}_{ab}G^b(w)}{z-w} + \ldots , \\
A^{\pm,i}(z)Q^a(w) &= \frac{\alpha^{\pm,i}_{ab}Q^b(w)}{z-w} + \ldots , \\
Q^a(z)G^b(w) &= \frac{2\alpha^{+,i}_{ab}A^{+,i}(w) - 2\alpha^{-,i}_{ab}A^{-,i}(w)}{z-w} + \frac{\delta^{ab}U(w)}{z-w} + \ldots , \\
U(z)G^a(w) &= \frac{Q^a(w)}{(z-w)^2} + \ldots .
\end{aligned} \tag{2.3}$$

$\alpha^{\pm,i}_{ab}$ here are 4×4 matrices, which project onto (anti)self-dual tensors. Explicitly,

$$\alpha^{\pm,i}_{ab} = \frac{1}{2}\Big(\pm\delta_{ia}\delta_{b0} \mp \delta_{ib}\delta_{a0} + \epsilon_{iab}\Big) . \tag{2.4}$$

They obey $SO(4)$ commutation relations:

$$[\alpha^{\pm,i},\alpha^{\pm,j}] = -\epsilon^{ijk}\alpha^{\pm k} \quad , \quad [\alpha^{+,i},\alpha^{-,j}] = 0 \quad , \quad \{\alpha^{\pm,i},\alpha^{\pm,j}\} = -\frac{1}{2}\delta^{ij} . \tag{2.5}$$

It is sometimes useful to employ spinor notation, where for instance $G^a \to G^{A\dot{A}} = \gamma^{A\dot{A}}_a G^a$ (and $\gamma^{A\dot{A}}_a$ are Dirac matrices); $A^{+,i} \to A^{AB} = \tau^{AB}_i A^{+,i}$ (where τ^i are Pauli matrices); $A^{-,i} \to A^{\dot{A}\dot{B}} = \tau^{\dot{A}\dot{B}}_i A^{-,i}$; and so on. An important subalgebra of $\mathcal{A}_\gamma$ is denoted $D(2,1|\alpha)$; here $\alpha = k^-/k^+ = \frac{\gamma}{1-\gamma}$. It is generated (in the NS sector) by L_0, $L_{\pm 1}$, $G^a_{\pm 1/2}$, and $A^{\pm,i}_0$.

2.2. Examples of large $\mathcal{N}=4$ SCFT's

The simplest example of a large $\mathcal{N}=4$ theory can be realized as a theory of a free

boson, ϕ, and four Majorana fermions, ψ_a, $a = 0, \ldots, 3$. Specifically, we have [26,25,12]:

$$
\begin{aligned}
T &= -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\psi^a \partial \psi^a \\
G^a &= -\frac{1}{6} i\epsilon^{abcd}\psi^b\psi^c\psi^d - i\psi^a\partial\phi \\
A^{\pm,i} &= \frac{i}{2}\alpha^{\pm,i}_{ab}\psi^a\psi^b \\
Q^a &= \psi^a \\
U &= i\partial\phi \; .
\end{aligned}
\tag{2.6}
$$

This theory was called the $\mathcal{T}_3$ theory in [27], but we shall herein use the notation $\mathcal{S}$ for simple.

The CFT $\mathcal{S}$ belongs to a family of large $\mathcal{N} = 4$ theories, labeled by a non-negative integer number κ [12]:

$$
\begin{aligned}
T &= -J^0 J^0 - \frac{J^a J^a}{\kappa + 2} - \partial\psi^a\psi^a \\
G^a &= 2J^0\psi^a + \frac{4}{\sqrt{\kappa+2}}\alpha^{+,i}_{ab} J^i \psi^b - \frac{2}{3\sqrt{\kappa+2}}\epsilon_{abcd}\psi^b\psi^c\psi^d \\
A^{-,i} &= \alpha^{-,i}_{ab}\psi^a\psi^b \\
A^{+,i} &= \alpha^{+,i}_{ab}\psi^a\psi^b + J^i \\
U &= -\sqrt{\kappa+2}J^0 \\
Q^a &= \sqrt{\kappa+2}\psi^a
\end{aligned}
\tag{2.7}
$$

where J^i denote $SU(2)$ currents at level κ and $J^0(z)J^0(w) \sim -\frac{1}{2}(z-w)^{-2}$. We shall denote these theories $\mathcal{S}_\kappa$. It is easy to check that (2.7) indeed generate the large $\mathcal{N} = 4$ algebra with $k^+ = \kappa + 1$ and $k^- = 1$. In fact, the $U(2)$ level κ theory of [12] admits *two* distinct large $\mathcal{N} = 4$ algebras. The second algebra is obtained by the outer automorphism and has $(k^+ = 1, k^- = \kappa + 1)$:

$$
\begin{aligned}
T &= -J^0 J^0 - \frac{J^a J^a}{\kappa + 2} - \partial\psi^a\psi^a \\
G^a &= 2J^0\psi^a + \frac{4}{\sqrt{\kappa+2}}\alpha^{+,i}_{ab} J^i \psi^b - \frac{2}{3\sqrt{\kappa+2}}\epsilon_{abcd}\psi^b\psi^c\psi^d \\
A^{-,i} &= \alpha^{-,i}_{ab}\psi^a\psi^b + J^i \\
A^{+,i} &= \alpha^{+,i}_{ab}\psi^a\psi^b \\
U &= +\sqrt{\kappa+2}J^0 \\
Q^a &= -\sqrt{\kappa+2}\psi^a
\end{aligned}
\tag{2.8}
$$

The $c = 3$ CFT $\mathcal{S} = \mathcal{S}_0$ appears as a special case, $\kappa = 0$.

Additional examples of large $\mathcal{N} = 4$ are provided by WZW coset models $\mathcal{W} \times U(1)$, where $\mathcal{W}$ is a gauged WZW model associated to a quaternionic (Wolf) space. Examples based on classical groups are $\mathcal{W} = G/H = \frac{SU(n)}{SU(n-2)\times U(1)}$, $\frac{SO(n)}{SO(n-4)\times SU(2)}$, and $\frac{Sp(2n)}{Sp(2n-2)}$. These theories carry large $\mathcal{N} = 4$ supersymmetry, with $k^+ = \kappa + 1$ and $k^- = \check{c}_G$; here κ is the level of the bosonic current algebra for the group G and $\check{c}_G$ its dual Coxeter number.

2.3. Unitary representations

The unitary representations of the superconformal algebra $\mathcal{A}_\gamma$ are labeled by the conformal dimension h, by the $SU(2)$ spins $\ell^\pm$, and by the $U(1)$ charge u. Character expansions can be found in appendix A. The generic *long* or *massive* representation has no null vectors under the raising operators of the algebra. On the other hand, the highest weight states $|\Omega\rangle_{\mathcal{A}_\gamma}$ of *short* or *massless* representations have the null vector [13]

$$\Big(G^{++\dot{+}}_{-1/2} - \frac{2u}{k^+ + k^-}\, Q^{++}_{-1/2} - \frac{2i(\ell^+ - \ell^-)}{k^+ + k^-}\, Q^{++}_{-1/2}\Big)|\Omega\rangle_{\mathcal{A}_\gamma} = 0\ . \tag{2.9}$$

(We have used the property that $|\Omega\rangle_{\mathcal{A}_\gamma}$ is a highest weight state for the $SU(2)$ current algebras.) Squaring this null vector leads to a relation among the spins $\ell^\pm$ and the conformal dimension h [13,28,15,17]

$$h_{\rm short} = \frac{1}{k^+ + k^-}\left(k^-\ell^+ + k^+\ell^- + (\ell^+ - \ell^-)^2 + u^2\right)\ . \tag{2.10}$$

Unitarity demands that all representations, short or long, lie at or above this bound: $h \geq h_{\rm short}$; and that the spins lie in the range $\ell^\pm = 0, \frac{1}{2}, ..., \frac{1}{2}(k^\pm - 1)$. When we consider $U(1)$ singlets, we shall denote representations by their labels (h, ℓ^+, ℓ^-); for short representations with $u = 0$ it is sufficient to specify them simply by (ℓ^+, ℓ^-). The conformal dimension of short representations is protected, as long as they do not combine into long ones.

3. A new index for theories with $\mathcal{A}_\gamma$ symmetry

In order to define a theory with $\mathcal{A}_\gamma$ symmetry we begin with a representation $\mathcal{H}_{RR}$ of the RR sector algebra $\mathcal{A}_{\gamma_{\rm left}} \oplus \mathcal{A}_{\gamma_{\rm right}}$. We then define the NS sector modules using

spectral flow.[1] The representation content of the theory is therefore summarized by the RR sector supercharacter:

$$Z(\tau,\omega_+,\omega_-;\overline{\tau},\tilde{\omega}_+,\tilde{\omega}_-) := \mathrm{Tr}_{\mathcal{H}_{RR}}\left[q^{L_0-c/24}\tilde{q}^{\tilde{L}_0-c/24} z_+^{2A_0^{+,3}}(-z_-)^{2A_0^{-,3}}\tilde{z}_+^{2\tilde{A}_0^{+,3}}(-\tilde{z}_-)^{2\tilde{A}_0^{-,3}}\right] . \tag{3.1}$$

Here and hereafter we denote $z_\pm = e^{2\pi i\omega_\pm}$ for left-movers and $\tilde{z}_\pm = e^{2\pi i\tilde{\omega}_\pm}$ for right-movers. We can take the ω's to be real. Sometimes we simply write Z for the partition function (3.1), or $Z(\mathcal{C})$ if we wish to emphasize dependence on the conformal field theory $\mathcal{C}$.

Now (3.1) can be expanded in the supercharacters of the irreducible representations, defined by

$$\mathrm{SCh}(\rho)(\tau,\omega_+,\omega_-) = \mathrm{Tr}_\rho q^{L_0-c/24} z_+^{2A_0^{+,3}} z_-^{2A_0^{-,3}}(-1)^{2A_0^{-,3}} \tag{3.2}$$

we just write $\mathrm{SCh}(\rho)$ when the arguments are understood. Explicit formulae for these characters have been derived by Peterson and Taormina and are reproduced in appendix B. Using the formulae of [15] one finds that short representations have a character with a first order zero at $z_+ = z_-$, while long representations have a character with a second order zero at $z_+ = z_-$.

Thanks to the second order vanishing of the characters of long representations we can define the *left-index* of the CFT $\mathcal{C}$ by

$$I_1(\mathcal{C}) := -z_+\frac{d}{dz_-}\bigg|_{z_-=z_+} Z\ . \tag{3.3}$$

Only short, or BPS representations can contribute on the left. On the right, long representations might contribute. However, due to the constraint $h-\overline{h}=0 \bmod 1$ the right-moving conformal weights which do contribute are rigid, and hence I_1 is a deformation invariant.

Of course, one could also define a right-index. Since we will consider left-right symmetric theories here this is redundant information. Nevertheless, it is often useful to define the left-right index:

$$I_2(\mathcal{C}) := z_+\tilde{z}_+\frac{d}{dz_-}\frac{d}{d\tilde{z}_-}Z \tag{3.4}$$

where one evaluates at $z_- = z_+$, $\tilde{z}_- = \tilde{z}_+$.

[1] Then, if one wishes, one can impose a GSO projection. Of course, the GSO projected theory is in general not a representation of $\mathcal{A}_{\gamma_{\mathrm{left}}}\oplus\mathcal{A}_{\gamma_{\mathrm{right}}}$.

Finally, using the explicit formulae of [15] it is not difficult to show that the contribution of the massless R sector rep (ℓ^+, ℓ^-, u) to the index is

$$-z_+ \frac{d}{dz_-}\Big|_{z_-=z_+} \mathrm{SCh}(\ell^+, \ell^-, u) = (-1)^{2\ell^-+1} q^{u^2/k} \Theta^-_{\mu,k}(\omega, \tau) \tag{3.5}$$

where $z = z_+ = \exp[2\pi i\omega]$, $\Theta^-_{\mu,k}$ is an odd level k-theta function (see appendix D) and

$$\mu = \mu(\rho) := 2(\ell^+ + \ell^-) - 1 \ . \tag{3.6}$$

We would like to make a number of remarks on this key result.

1. When the highest weight state of a long representation saturates the BPS bound the representation decomposes as a sum of two short representations. The formula relating massive and massless characters when h satisfies the unitarity bound is ([15] eq. 5.7)
$$\mathrm{Ch}(\ell^+, \ell^-) + \mathrm{Ch}(\ell^+ - \frac{1}{2}, \ell^- + \frac{1}{2}) = \mathrm{Ch}_{\mathrm{massive}}(h, \ell^+, \ell^- + \frac{1}{2}) \tag{3.7}$$
and it is a nice check that the sum of the indices vanishes. If the index vanishes on a linear combination of characters then using (3.7) that combination can be written as an integral linear combination (with signs) of massive characters.
2. The unitarity bounds on spins in the R sector are $1/2 \le \ell^\pm \le \frac{1}{2}k^\pm$. In (3.5) we have odd level k theta functions. There are $k-1$ linearly independent such odd functions. For the unitary range, μ takes precisely the values $1, \dots, k-1$.
3. The only states which can contribute to the index have
$$2(A^{+,3} + A^{-,3}) = \pm\mu(\rho) \bmod 2k \tag{3.8}$$
$$L_0 - c/24 = \frac{u^2}{k} + \frac{1}{k}\Big[(A^{+,3} + A^{-,3})^2\Big] \ . \tag{3.9}$$
(This might appear confusing since the BPS bound is expressed in terms of representation labels in terms of $\frac{1}{k}(\ell^+ + \ell^- - 1/2)^2$. The point is that there is not a unique highest weight vector amongst the R-sector groundstates, and the representation labels (ℓ^+, ℓ^-) do not label the quantum numbers of any particular state. See the character formulae in appendices A and B.)
4. The fact that all characters vanish at $z_+ = z_-$ is related to the existence of the simple free field theory $\mathcal{S}$ with $\mathcal{A}_\gamma$ symmetry, defined in equation (2.6). It has $k^+ = k^- = 1$ and hence $c = 3$. Note that the free $\mathcal{S}$-theory defined by $U, Q^{A\dot{A}}$ is always present as a subsector of any theory with $\mathcal{A}_\gamma$ symmetry. One can always make a "GKO coset construction" factoring out the $\mathcal{S}$ theory and leaving a W-algebra type symmetry $\tilde{\mathcal{A}}_\gamma$. The characters of the quotient $\tilde{\mathcal{A}}_\gamma$ W-algebra are nonvanishing for short representations, and have a first order zero for long representations.

3.1. Constraints on the spectrum

In general we expect that $\mathcal{H}_{RR}$ is completely decomposable in irreducible $\mathcal{A}_{\gamma_{\rm left}} \oplus \mathcal{A}_{\gamma_{\rm right}}$ representations, i.e.

$$\mathcal{H}_{RR} = \bigoplus_{\rho,\tilde{\rho}} N(\rho,\tilde{\rho})\rho \otimes \tilde{\rho} \tag{3.10}$$

with nonnegative integer degeneracies $N(\rho,\tilde{\rho})$. We now describe what the index can teach us about the degeneracies $N(\rho,\tilde{\rho})$.

The left-index is an expression of the form:

$$I_1 = \sum_{\mu=1}^{k-1} \sum_{\Gamma} \Theta^-_{\mu,k}(\omega,\tau)\overline{\Xi_{\mu,u,\tilde{u}}(\tau,\tilde{\omega}_+,\tilde{\omega}_-)}q^{u^2/k}\overline{q}^{\tilde{u}^2/k} \tag{3.11}$$

where $\Gamma = \{(u,\tilde{u})\}$ is the set of $U(1)\times\widetilde{U(1)}$ charges. Now, $\overline{\Xi_\mu}$ has contributions from both the short and long representations:

$$\begin{aligned}\Xi_{\mu,u,\tilde{u}} = & \sum_{\substack{\ell^+ + \ell^- = (\mu+1)/2 \\ \tilde{\ell}^+,\tilde{\ell}^-}} (-1)^{2\ell^-+1} n(\ell^+,\ell^-,u;\tilde{\ell}^+,\tilde{\ell}^-,\tilde{u}) \ \mathrm{SCh}_{(\tilde{\ell}^+,\tilde{\ell}^-)}(\tau,\tilde{\omega}_+,\tilde{\omega}_-) + \\ & + \sum_{\substack{\ell^+ + \ell^- = (\mu+1)/2 \\ \tilde{\rho}:\mathrm{long}}} (-1)^{2\ell^-+1} N(\ell^+,\ell^-,u;\tilde{\rho}) \ \mathrm{SCh}_{\tilde{\rho}}(\tau,\tilde{\omega}_+,\tilde{\omega}_-)\end{aligned} \tag{3.12}$$

where the first line contains the sum over short representations on both left and right, and the second line contains the sum over (*short*, *long*) representations (with fixed $U(1)$ charge $\tilde{u}$). For convenience, we denoted the degeneracies of the short representations by $n(\ell^\pm,u;\tilde{\ell}^\pm,\tilde{u}) := N(\ell^\pm,u;\tilde{\ell}^\pm,\tilde{u})$. Similarly, the left-right index I_2 knows only about the short representations. Specifically, it has the form

$$I_2 = \sum_{\mu,\tilde{\mu}=1}^{k-1} \sum_{\Gamma} D(\mu,u;\tilde{\mu},\tilde{u})\Theta_{\mu,k}\,\overline{\Theta_{\tilde{\mu},k}}\,q^{u^2/k}\,\overline{q}^{\tilde{u}^2/k} \tag{3.13}$$

with

$$D(\mu,u;\tilde{\mu},\tilde{u}) = \sum n(\ell^\pm,u;\tilde{\ell}^\pm,\tilde{u})(-1)^{2\ell^-+2\tilde{\ell}^-}\delta_{\mu,2(\ell^++\ell^-)-1,\ \mathrm{mod}\ 2k}\,\delta_{\tilde{\mu},2(\tilde{\ell}^++\tilde{\ell}^-)-1,\ \mathrm{mod}\ 2k} \tag{3.14}$$

The multiplicities of the representations that occur in the decomposition of I_1 and I_2 can be both positive and negative integers. These multiplicities are related to $N(\rho,\tilde{\rho}) \geq 0$ via (3.12) and (3.14), respectively.

3.2. Constraints of modular invariance

The partition function of the theory in the odd spin structure is the supercharacter. This should be "modular invariant" in the sense that it satisfies the transformation rule:

$$Z_{RR}\Big(\frac{a\tau+b}{c\tau+d},\frac{\omega_+}{c\tau+d},\frac{\omega_-}{c\tau+d};\dots\Big)=e^{\Phi}Z_{RR}(\tau,\omega_+,\omega_-;\overline{\tau},\tilde{\omega}_+,\tilde{\omega}_-) \tag{3.15}$$

where

$$\Phi=2\pi i\Bigg(k^+\frac{c\omega_+^2}{c\tau+d}+k^-\frac{c\omega_-^2}{c\tau+d}\Bigg)-2\pi i\Bigg(k^+\frac{c\tilde{\omega}_+^2}{c\overline{\tau}+d}+k^-\frac{c\tilde{\omega}_-^2}{c\overline{\tau}+d}\Bigg) \tag{3.16}$$

The phase comes from the singular term in the JJ ope. The transformation rule (3.15) shows that Z_{RR} is a Jacobi form of weight $(0;0)$ and index $(k,k;k,k)$ [29].

The modular transformation rule (3.15) translates into a modular property of the index I_2:

$$I_2(\frac{a\tau+b}{c\tau+d},\frac{\omega}{c\tau+d},\frac{\tilde{\omega}}{c\overline{\tau}+d})=(c\tau+d)(c\overline{\tau}+d)e^{\Phi}I_2(\tau,\omega,\tilde{\omega}) \tag{3.17}$$

where now the phase is

$$\Phi=2\pi i\Big(k\frac{c\omega^2}{c\tau+d}\Big)-2\pi i\Big(k\frac{c\tilde{\omega}^2}{c\overline{\tau}+d}\Big) \tag{3.18}$$

We now show how the modular covariance (3.17) of the index I_2 together with the simple transformation properties of level k theta functions can be used to constrain the spectrum. As a simple example suppose that the degeneracies have the property:

$$n(\ell^+,\ell^-,u;\tilde{\ell}^+,\tilde{\ell}^-,\tilde{u})=n(\ell^+,\ell^-;\tilde{\ell}^+,\tilde{\ell}^-)n(u,\tilde{u}) \tag{3.19}$$

and suppose moreover that $n(u,\tilde{u})$ is such that

$$Z_\Gamma=\sum_{u,\tilde{u}}n(u,\tilde{u})q^{u^2/k}\,\overline{q}^{\tilde{u}^2/k} \tag{3.20}$$

is a Siegel-Narain theta function, i.e. transforms as a modular form of weight $(1/2,1/2)$. In this case, using the modular transformation rules for level k theta functions in Appendix D, we see that (3.17) implies that

$$\sum_{\mu,\tilde{\mu}=1}^{k-1}D(\mu,\tilde{\mu})\chi^{(k-2)}_{(\mu-1)/2}\,\overline{\chi^{(k-2)}_{(\tilde{\mu}-1)/2}} \tag{3.21}$$

must define a modular invariant partition function for the bosonic $SU(2)$ WZW model at level $k-2$. We use $\chi_j^{(r)}$ to denote the character of the affine level r $SU(2)$ representation with spin j, thus

$$\chi_j^{(r)}(\omega,\tau) = \frac{\Theta^-_{2j+1,r+2}(\omega,\tau)}{\Theta^-_{1,2}(\omega,\tau)}. \tag{3.22}$$

The constraint (3.21) on the spectrum of theories with $\mathcal{A}_\gamma$ symmetry has been found before in ref. [16]. This constraint relies crucially on (3.19). As we will see in the example considered below, the constraint (3.19) on the spectrum doesn't apply to certain cases of interest. Note that for a modular form of nonzero weight (such as Z_Γ) the constant term $\sim q^0\overline{q}^0$ is *not* modular invariant, so in theories where (3.19) does not hold the above constraints do not apply to the spins even in the charge zero sector.

A more general class of theories for which there are simple constraints on the spectrum from modular invariance can be easily described by extending the definition (3.1) to include an extra operator keeping track of the $U(1)$ charges of the states. Thus we define:

$$Z(\cdots,\chi_L,\chi_R) := e^{\frac{k\pi}{4\tau_2}(\chi_L^2+\chi_R^2)}\mathrm{Tr}_{\mathcal{H}_{RR}}\left(\cdots e^{2\pi(\chi_L U_0 - \chi_R \tilde{U}_0)}\right) \tag{3.23}$$

The exponential prefactor is included so that Z has nice modular transformation properties (i.e. without the phase in (3.15)), provided we transform

$$(\chi_L,\chi_R) \to \left(\frac{\chi_L}{c\tau+d},\frac{\chi_R}{c\overline{\tau}+d}\right).$$

From (3.23) we obtain an index $I_2(\mathcal{C},\chi)$. Suppose that the $U(1)$ spectrum is such that we have

$$I_2(\mathcal{C},\chi) = \sum_{\beta\in\Lambda^*/\Lambda} I_2^\beta \Theta_\Lambda(\tau,0,\beta;P;\sqrt{k/2}\chi) \tag{3.24}$$

where I_2^β are χ-independent, and Θ_Λ is a Siegel-Narain theta function for a lattice $\Lambda \cong \sqrt{k/2}II^{1,1}$. (See Appendix E for our conventions on Siegel-Narain theta functions.) A nontrivial example of such a theory may be found in sec. 4.6 below. If k is even, Θ_Λ transforms in a simple representation of the modular group. On the other hand I_2^β can be written in terms of level k theta functions of $\omega,\tilde{\omega}$, which also transform simply and we obtain a simple set of constraints on the spectrum. The main example computed below, that of $\mathrm{Sym}^N(\mathcal{S})$, is in this class of theories.

3.2.1. *Gauging the $U(1)$*

Given a theory $\mathcal{C}$ with $\mathcal{A}_\gamma$ symmetry one can in principle gauge a $U(1)$ subalgebra to produce a theory $\hat{\mathcal{C}}$ with $\tilde{\mathcal{A}}_\gamma$ symmetry. (This is quite similar to the way one derives the parafermion theories from the $SU(2)$ WZW model.) Conversely, given a theory $\hat{\mathcal{C}}$ with $\tilde{\mathcal{A}}_\gamma$ symmetry we one can always tensor with the basic $\mathcal{A}_\gamma$ theory, $\mathcal{S}$, to produce a new theory, $\check{\mathcal{C}}$ with $\mathcal{A}_\gamma$ symmetry. We thus have a transformation of theories $\mathcal{C} \to \check{\mathcal{C}}$ with $\mathcal{A}_\gamma$ symmetry.

We will now show that it is possible in some cases to describe how the index $I_2(\mathcal{C})$ is related to that of $I_2(\check{\mathcal{C}})$. We will limit ourselves to theories which are rational with respect to $\mathcal{A}_\gamma$, and which satisfy the criterion (3.24).

First, let us describe the gauging process. We can gauge the axial $U(z) + \tilde{U}(\overline{z})$ or the vector $U(z) - \tilde{U}(\overline{z})$ symmetry. For definiteness, let us gauge the vector symmetry. Then gauging the symmetry projects onto vectors $\psi \in \mathcal{H}(\mathcal{C})$ satisfying:

$$
\begin{aligned}
U_n \psi &= 0 \qquad\qquad n > 0 \\
\tilde{U}_n \psi &= 0 \qquad\qquad n > 0 \\
(U_0 - \tilde{U}_0)\psi &= 0
\end{aligned}
\tag{3.25}
$$

The subspace satisfying (3.25) is a representation of $\tilde{\mathcal{A}}_\gamma$. However, from the decomposition of characters $\mathrm{SCh}^{\mathcal{A}_\gamma} = \mathrm{SCh}^{\tilde{\mathcal{A}}_\gamma}\mathrm{SCh}^{\mathcal{S}}$ it is clear that the theory is infinitely degenerate with respect to $\tilde{\mathcal{A}}_\gamma$, and we wish to have a theory which is rational with respect to $\tilde{\mathcal{A}}_\gamma$. Therefore we must also impose

$$
Q_r^{A\dot{B}} \psi = 0 \qquad\qquad r > 0 \tag{3.26}
$$

This theory is still infinitely degenerate because there will be infinitely many primaries with $u_L = u_R$. However, if we consider the collection of charge vectors:

$$
\left\{ \sqrt{\tfrac{k}{2}} (u_L; u_R) \right\} \subset \mathrm{I\!R}^{1,1} \tag{3.27}
$$

where $(p_L; p_R) \in \mathrm{I\!R}^{1,1}$ has metric $p_L^2 - p_R^2$, then we can consider the case where these values lie in a lattice Λ^*, where Λ is an even integral lattice isomorphic to $\sqrt{\frac{k}{2}} II^{1,1}$. Now, we can identify states whose charge vectors differ by a vector in Λ. In this way we produce the gauged theory $\hat{\mathcal{C}}$ where $\hat{\mathcal{C}}$ which is a rational $\tilde{\mathcal{A}}_\gamma$ theory.

The relation of the index of $\mathcal{C}$ to that of $\hat{\mathcal{C}} \otimes \mathcal{S}$ is easily stated. We decompose I_2 according to (3.24). Then

$$I_2(\hat{\mathcal{C}} \otimes \mathcal{S}) = \Big(\sum_{\beta_L = \beta_R} I_2^{\beta} \Big) I_2(\mathcal{S}) \tag{3.28}$$

One can check that the right hand side of (3.28) has the correct modular transformation properties using the transformation properties of Siegel-Narain theta functions of higher level.

3.3. Relation to the $\mathcal{N} = 2$ elliptic genus

In $\mathcal{N} = 2$ theory the procedure analogous to the operation (3.3) is to evaluate the character

$$\mathrm{Tr}_R \big[q^{L_0 - c/24} z^{J_0} \big] \tag{3.29}$$

and set $z = -1$. When applied to the left-moving sector this defines the elliptic genus of the theory.[2] In this case, only the states with $L_0 = c/24$ can contribute, and hence the index is an integer. It counts the R ground states. Similar remarks hold for the small $\mathcal{N} = 4$ algebra. In our case, because of the larger $\mathcal{A}_\gamma$ symmetry we are able to keep z generic and yet maintain independence under deformation of parameters. The states which are contributing are not BPS with respect to the $\mathcal{N} = 2$ subalgebras, but they are nevertheless protected by the large $\mathcal{N} = 4$ symmetry. That is why the $\mathcal{A}_\gamma$-index is valued in the ring of level k theta functions, rather than integers.

Note that, in striking contrast to $\mathcal{N} = 2$ and small $\mathcal{N} = 4$ theories $h - c/24$ is typically *positive* for Ramond groundstates in theories with $\mathcal{A}_\gamma$ symmetry. Indeed, note that the BPS bound in the R sector can be written $h - c/24 \geq \mu(\rho)^2/4k > 0$. This has interesting implications, discussed below, for the geometrical interpretation of the index.

The $\mathcal{A}_\gamma$ algebra has $\mathcal{N} = 2$ subalgebras. These are studied in [30], so it is interesting to ask about how the index is related to the elliptic genus with respect to these $\mathcal{N} = 2$ subalgebras. Up to an $SU(2) \times SU(2)$ transformation we can consider the generators

$$\begin{aligned}
\mathcal{G}_m^+ &= i\sqrt{2} G_m^{\dot{+},+} \\
\mathcal{G}_m^- &= i\sqrt{2} G_m^{\dot{-},-} \\
J_m &= 2i\big(\gamma A_m^{+,3} - (1-\gamma) A_n^{-,3}\big)
\end{aligned} \tag{3.30}$$

[2] Thus, the contribution of the right-movers gives a holomorphic function of $\overline{q}, \tilde{z}$. Throughout this section we suppress mention of the right-moving sector for simplicity.

Together with L_n, these generate an $\mathcal{N}=2$ subalgebra of $\mathcal{A}_\gamma$ with

$$\begin{aligned}
\{\mathcal{G}^+_m, \mathcal{G}^-_n\} &= 2L_{n+m} + (m-n)J_{m+n} + \frac{c}{12}\delta_{n+m,0}(4m^2-1) \\
[J_m, \mathcal{G}^+_n] &= -\mathcal{G}^+_{n+m} \\
[J_m, \mathcal{G}^-_n] &= +\mathcal{G}^+_{n+m}
\end{aligned} \tag{3.31}$$

If Φ^{m^+,m^-} is an NS-sector BPS multiplet of $\mathcal{A}_\gamma$ with $kh = k^+\ell^- + k^-\ell^+ + (\ell^+ - \ell^-)^2$ then a computation of the charge under J_0 shows that we do not get chiral primaries under the $\mathcal{N}=2$ subalgebra unless $\ell^+ = \ell^- = \ell$ in which case $\Phi^{\ell,-\ell}$ is anti-chiral primary and $\Phi^{-\ell,\ell}$ is chiral primary.

It is interesting to consider spectral flow using this $\mathcal{N}=2$ subalgebra. $Q^{\dot{+},+}_n, Q^{\dot{-},-}_n$ have charges $-1,+1$ while

$$\begin{aligned}
[J_m, Q^{\dot{+},-}_n] &= (1-2\gamma)Q^{\dot{+},-}_{n+m} \\
[J_m, Q^{\dot{-},+}_n] &= -(1-2\gamma)Q^{\dot{-},+}_{n+m}
\end{aligned} \tag{3.32}$$

Note that $\gamma = \frac{1}{2}$ is an interesting special case. Then spectral flow by 1 unit takes the NS sector to the R sector, but (3.32) remain in the NS sector. Thus, one must be careful about referring to NS and R sectors since they can mean different things for the $\mathcal{A}_\gamma$ algebra and the $\mathcal{N}=2$ subalgebra.

It might appear that one can introduce a new index using this $\mathcal{N}=2$ subalgebra. In fact, the index turns out to be already encoded in the index we have already introduced. To be more precise, let us suppose that the NS sector coincides for the $\mathcal{A}_\gamma$ and $\mathcal{N}=2$ algebra. The $\mathcal{N}=2$ index can be expressed in terms of the NS-sector trace

$$\mathrm{Tr}_{\mathcal{H}_{NS}}\left[e^{2\pi i\tau(L_0 - \frac{1}{2}J_0)}e^{i\pi J_0}\right] \tag{3.33}$$

Now using $\mathcal{A}_\gamma$ spectral flow we find that (3.33) is, up to a simple prefactor the trace (3.2) evaluated at

$$\omega_+ = \omega_- = \frac{1}{2}(\tau+1)(1-\gamma) \tag{3.34}$$

We may conclude two things from this: First, the $\mathcal{N}=2$ index in fact vanishes, since R-sector $\mathcal{A}_\gamma$ characters always vanish when $\omega_+ = \omega_-$. Second, the index (3.3) is a generalization of the "new index" $\mathrm{Tr} q^{L_0 - c/24}F(-1)^F$ of [31].

3.4. An index for the Wigner contraction $\mathcal{A}_{k^+,\infty}$

A Wigner contraction of $\mathcal{A}_{k^+,k^-}$ produces an extension of a small $\mathcal{N}=4$ algebra. This contraction is relevant to the study of holography for strings on $AdS_3 \times \mathbf{S}^3 \times T^4$. In this context ref. [22] introduced an index for computing BPS states. In this section we clarify the relation of the index of [22] to the present index.

We define the contraction by separating the zeromodes of the $SU(2)^-$ current algebra and defining

$$\begin{aligned}\tilde{A}_n^{-,i} &= \frac{1}{\sqrt{k^-}} A_n^{-,i} \qquad n \neq 0\\ \tilde{A}_0^{-,i} &= A_0^{-,i},\\ \tilde{U} &= \frac{1}{\sqrt{k^-}} U,\\ \tilde{Q}^a &= \frac{1}{\sqrt{k^-}} Q^a\end{aligned} \tag{3.35}$$

We then take the limit $k^- \to \infty$ in the commutation relations, holding $\tilde{A}^{-,i}, \tilde{U}, \tilde{Q}^a$ fixed to obtain the small $\mathcal{N}=4$ algebra extended by a $U(1)^4$ current algebra *together with* a global, finite-dimensional $SU(2)^-$ algebra spanned by $A_0^{-,i}$. The $U(1)^4$ currents form a $\mathbf{3} \oplus \mathbf{1}$ and the fermionic partners are in the $\mathbf{2} \oplus \mathbf{2}$. If desired, one could make a further contraction of the global $SU(2)^-$ algebra to $U(1)^3$, thus, supplying the remaining zeromodes of the $U(1)^4$ current algebra.

The key remark is the observation of [14,15] that the characters for $\mathcal{A}_\gamma$ have well-defined limits for $k^- \to \infty$, holding all other quantum numbers fixed. By inspection one finds that the massless characters have the form:

$$\mathrm{SCh}_0^{\mathcal{A}_\gamma,R} = (z_+ + z_+^{-1} - z_- - z_-^{-1}) G_0(k^+, k^-, q, z_+, z_-) \tag{3.36}$$

while the massive characters have the form

$$\mathrm{SCh}_m^{\mathcal{A}_\gamma,R} = (z_+ + z_+^{-1} - z_- - z_-^{-1})^2 G_m(k^+, k^-, q, z_+, z_-) \tag{3.37}$$

where the functions G_0 and G_m have the following properties: First, these functions have smooth limits for $k^- \to \infty$. Second, the functions are nonzero at $z_+ = z_-$ and do not get extra zeroes either at $z_+ = z_- = 1$ or as $k^- \to \infty$.

Now working in the limit $k^- = \infty$, ref. [22] sets $z_- = 1$. Notice that in this case the prefactor

$$(z_+ + z_+^{-1} - z_- - z_-^{-1}) = z_+^{-1}(z_+ - 1)^2 \tag{3.38}$$

develops a second order zero at $z_+ = 1$. From (3.36)(3.37) it is clear that to form a nonzero index one must take two derivatives with respect to z_+ and set $z_+ = 1$. This is the index of [22]. The precise relation between the indices is that

$$\frac{d}{dz}\Big|_{z=1} I_1(\mathcal{C}) \tag{3.39}$$

is, up to a simple numerical factor, identical with the index of [22]. At finite k^- the RHS of (3.39) is a function of $q, \overline{q}, \tilde{z}_+, \tilde{z}_-$. As $k^- \to \infty$ the q-dependence drops out and the expression becomes a holomorphic function of $\overline{q}, \tilde{z}_+$ at $\tilde{z}_- = 1$.

In hindsight, the index of [22] was unnecessarily restrictive. Since the small $\mathcal{N} = 4$ algebra has a global $SU(2)^-$ symmetry one could have kept $z_- \neq 1$, and used the index described in the present paper. Thus, it might be of interest to reexamine the partition function in the $AdS_3 \times \mathbf{S}^3 \times T^4$ background using the more powerful index I_1. A word of warning is in order here. In the limit $k^- \to \infty$, we have $\mathbf{S}^3 \times \mathbf{S}^1 \to \mathbb{R}^3 \times \mathbf{S}^1$. The zeromode algebra $A_0^{-,i}, U_0$ is broken by the identifications one would want to make to produce T^4. Note, in this connection, that the argument of [22] for invariance of the index fails precisely for states with nonzero $U(1)$ charges (see remarks under eq. 3.11 of [22]). Thus, the index of [22] only applies to states where we could replace $T^4 \to \mathbb{R}^4$, and this coincides with the $k^- \to \infty$ limit of $\mathbf{S}^3 \times \mathbf{S}^1$ in the charge zero sector.

3.5. Geometrical interpretation of the index

In this section we would like to call attention to some interesting open problems concerning the geometrical interpretation of the indices I_1, I_2 defined above in the case of a supersymmetric nonlinear sigma model with target space X having $\mathcal{A}_{\gamma_{\rm left}} \oplus \mathcal{A}_{\gamma_{\rm right}}$ symmetry.

In $\mathcal{N} = 2$ sigma models, with a Kahler target space X, the elliptic genus $\chi(q, y)$ computes topological invariants of the space. The $q \to 0$ limit gives the character [11]:

$$\sum_{r,s=0}^{\dim X} (-1)^{r+s} h^{r,s}(X) y^{s-\frac{1}{2}\dim_c X} \tag{3.40}$$

The same is true for small $\mathcal{N} = 4$ sigma models with hyperkahler target space.

Moreover, if one does not take the $q \to 0$ limit then the elliptic genus $\chi(q, y)$ has the interpretation of a.) computing an infinite set of indices of Dirac operators coupled

to bundles on X and b.) more conceptually, is the $U(1)$-equivariant index of the Dirac operator on the loop space of X [1,2,3,4,5,6,7,8,9,10,11].

It might be very interesting to generalize these statements to the large $\mathcal{N}=4$ indices I_1, I_2. The reason is that, as we have remarked above $h-c/24$ is positive in the R sector. From the geometrical point of view the reason for this is that for fixed $k^{\pm}$ the "size" of the target space is fixed. That is, one cannot, at *fixed* $k^{\pm}$, take an $\alpha' \to 0$ limit of the target space and recover classical geometry. For this reason, the index for $\mathcal{A}_\gamma$ theories is "more stringy" than its $\mathcal{N}=2$ and small $\mathcal{N}=4$ counterparts. This is closely related to the fact that the index is a theta function, rather than an integer.

In order to address this problem it is necessary to clarify the geometrical conditions on a σ-model such that it has $\mathcal{A}_\gamma$ symmetry. References [17,32] have some interesting relevant material for this problem, but the precise statement does not appear to be available. Next one would wish to find a geometrical interpretation of the BPS condition [13]

$$(\tilde{G}_{-\frac{1}{2}})^{(A_1}_{(\dot{A}_1}\Phi^{A_2\cdots A_n)}_{\dot{A}_2\cdots\dot{A}_m)} = 0 \tag{3.41}$$

in terms of differential equations on some tensors on the target space.

4. Application to symmetric product theories

The elliptic genus of symmetric product theories has proven to be quite useful in investigations of the AdS/CFT correspondence in the case of two-dimensional conformal field theories [20,21,22,23,24]. It is therefore natural to devote some special attention to symmetric product theories with $\mathcal{A}_\gamma$ symmetry.

Suppose a CFT $\mathcal{C}_0$ has $\mathcal{A}_\gamma$ symmetry. We would like to define a new theory, the symmetric product orbifold, with $\mathcal{A}_\gamma$ symmetry. We define the theory by applying the symmetric product construction to $\mathcal{H}_{RR}$, and then construct the remaining sectors by spectral flow. The Hilbert space in the RR sector has the form:

$$\mathcal{H}_{RR}(\mathrm{Sym}^N(\mathcal{C}_0)) = \bigoplus_{(n)^{\ell_n}} \bigotimes_n \mathrm{Sym}^{\ell_n}(\mathcal{H}^{(n)}_{RR}(\mathcal{C}_0)) \tag{4.1}$$

where $\mathcal{H}^{(n)}_{RR}(\mathcal{C}_0)$ is the twisted sector corresponding to a cycle of length n and we sum over partitions $\sum n\ell_n = N$, in the standard way. In order for the Hilbert space (4.1) to be a representation of the R-moded $\mathcal{A}_\gamma$ algebra it is quite important that we take a *graded*

tensor product, relative to the $\mathbb{Z}_2$-grading of $(-1)^F$. With this understood (4.1) can be used to construct an $\mathcal{A}_\gamma$-invariant theory.

In [19] a procedure was given for expressing the partition function of symmetric product theories in terms of that of the parent theory. The argument is reviewed briefly in appendix F. There are two ways of stating the result. One involves infinite products and the other involves Hecke operators. For purposes of computing the index it is more useful to give the formulation in terms of Hecke operators. We introduce the generating functional:

$$\mathcal{Z} := 1 + \sum_{N\geq 1} p^N \mathrm{STr}_{\mathrm{Sym}^N(\mathcal{C}_0)}\Big[q^{L_0 - c/24}\tilde{q}^{\tilde{L}_0 - c/24} z_+^{2A_0^{+,3}} z_-^{2A_0^{-,3}} \tilde{z}_+^{2\tilde{A}_0^{+,3}} \tilde{z}_-^{2\tilde{A}_0^{-,3}}\Big] \tag{4.2}$$

Then the result of appendix F is:

$$\log \mathcal{Z} = \sum_{M=1}^{\infty} p^M T_M Z_0 \tag{4.3}$$

where Z_0 is the partition function of $\mathcal{C}_0$ and the Hecke operator is defined by

$$T_M Z_0 := \frac{1}{M} \sum_{M=ad} \sum_{b=0}^{d-1} Z_0\Big(\frac{a\tau + b}{d}, a\omega_+, a\omega_-; \frac{a\overline{\tau} + b}{d}, a\tilde{\omega}_+, a\tilde{\omega}_-\Big) \tag{4.4}$$

The first sum is over all factorizations of M into a product of integers $a \times d$.

Now let us compute the index. The coefficient Z_N of p^N in (4.2) will be a polynomial in the Hecke transforms $T_M Z_0$:

$$Z_N = T_N Z_0 + \cdots + \frac{1}{N!}(T_1 Z_0)^N \tag{4.5}$$

All summands in (4.5) beyond the leading term are products of more than one Hecke transform. Since $T_M Z = 0$ for $z_- = z_+$ the only term which survives the index operation is the linear term. Thus the index simplifies enormously compared to the full partition function and we have

$$I_1(\mathrm{Sym}^N(\mathcal{C}_0)) = -z_+ \frac{d}{dz_-} T_N Z_0 \tag{4.6}$$

evaluated at $z_- = z_+ = z$.

4.1. Interpreting the states which contribute to the index

We would like to make three remarks on the interpretation of various terms in the index.

First, we make a distinction between *short-string* and *long-string* contributions. The Hecke operator $T_N Z_0$ always involves two kinds of terms, $a = N, d = 1$ and $a = 1, d = N$. (When N is prime these are the only contributions, but in general there are other contributions.) If we view the trace as the partition function of the conformal field theory on a torus, then these two contributions have natural interpretations in terms of covering tori. The $a = N, d = 1$ term corresponds to strings which are "long strings" in the Euclidean time direction. The $a = 1, d = N$ term corresponds to strings which are "long strings" in the space direction. In general, when N is not prime we have a combination of strings which are d-long in the space direction, wrapping $N/d = a$ times in the Euclidean time direction. In the supergravity interpretation of the holographic dual to $\mathrm{Sym}^N(\mathcal{C}_0)$ it turns out that the $a = N, d = 1$ term corresponds to the contribution of supergravity particles to the index while the $a = 1, d = N$ term presumably comes from other geometries, possibly conical defect geometries [33].

Secondly, we would like to discuss the contributions of specified twisted sectors in terms of Hecke operators. In order to associate contributions of particular states with various terms in the index we introduce formal variables x_n to keep track of cycles of length n and we multiply $\mathrm{Sym}^{\ell_n}(\mathcal{H}^{(n)}(\mathcal{C}_0))$ by $x_n^{\ell_n}$. If one follows through the derivation in appendix F one discovers that we have the modified Hecke operator:

$$\hat{T}_M Z_0 := \frac{1}{M} \sum_{M=ad} \sum_{b=0}^{d-1} (x_d)^a Z_0 \Big(\frac{a\tau + b}{d}, a\omega_+, a\omega_-; \frac{a\overline{\tau} + b}{d}, a\tilde{\omega}_+, a\tilde{\omega}_- \Big) \tag{4.7}$$

Using this expression in (4.5) and collecting terms proportional to a given monomial $\prod_n x_n^{\ell_n}$ gives an expression for the contribution of that twisted sector to the full partition function. It is then clear that the contribution to (4.6) from the $N = ad$ term in (4.4) comes entirely from states which are in the twisted sector:

$$\mathrm{Sym}^a\big(\mathcal{H}^{(d)}(\mathcal{C}_0)\big) \tag{4.8}$$

in the sum (4.1).

Finally, we would like to mention the *inheritance principle.* This is a useful observation for interpreting which states contribute to the index. Recall from (3.8) - (3.9) that the states which contribute to the index satisfy

$$k(L_0 - c/24) = U^2 + (A_0^{+,3} + A_0^{-,3})^2 \tag{4.9}$$

If there is a state in $\mathcal{C}_0$ which saturates the bound, then the corresponding state in the long string sector of $\mathrm{Sym}^N(\mathcal{C}_0)$ will also saturate the bound, since $k \to Nk$ while the eigenvalue Δ of $L_0 - c/24$ goes to Δ/N and the eigenvalues of $A_0^{+,3}$ and $A_0^{-,3}$ remain unchanged.

4.2. The Hecke algebra

The Hecke operators defined in (4.4) satisfy a beautiful algebra.[3] In order to describe this algebra, let us consider operators acting on the space of functions $f(\tau,\omega)$ such that $f(\tau+1,\omega) = f(\tau,\omega)$. For any positive integer n let us introduce the operators

$$\begin{aligned}
(U_n f)(\tau,\omega) &:= \frac{1}{n}\sum_{b=0}^{n-1} f\big(\frac{\tau+b}{n},\omega\big) \\
(V_n f)(\tau,\omega) &:= \frac{1}{n} f(n\tau, n\omega) \\
(W_n f)(\tau,\omega) &:= \frac{1}{n} f(\tau, n\omega)
\end{aligned} \tag{4.10}$$

Note that we obviously have

$$\begin{aligned}
V_{n_1}V_{n_2} &= V_{n_2}V_{n_1} = V_{n_1 n_2} \\
W_{n_1}W_{n_2} &= W_{n_2}W_{n_1} = W_{n_1 n_2} \\
V_{n_1}W_{n_2} &= W_{n_2}V_{n_1}
\end{aligned} \tag{4.11}$$

Slightly less obvious is

$$U_{n_1}U_{n_2} = U_{n_2}U_{n_1} = U_{n_1 n_2} \tag{4.12}$$

but this follows from noting that if $b_i \in \{0,\dots,n_i-1\}$ then $(b_1 + n_1 b_2) \bmod n_1 n_2$ takes all values in $\{0,\dots,n_1 n_2 - 1\}$.

The algebra satisfied by U_n and V_n is a little more intricate. First, if $(n_1,n_2)=1$ then

$$V_{n_1}U_{n_2} = U_{n_2}V_{n_1} \tag{4.13}$$

[3] For some material on Hecke theory for Jacobi forms see the book of Eichler and Zagier [29].

(this uses $f(\tau+1,\omega)=f(\tau,\omega)$). On the other hand, V_n and U_n do *not* commute. More generally, V_{n_1} and U_{n_2} do not commute if $(n_1,n_2)>1$. An easy computation shows

$$U_n V_n = \frac{1}{n} W_n \tag{4.14}$$

while

$$(V_n U_n f)(\tau,\omega) = \frac{1}{n^2}\sum_{b=0}^{n-1} f(\tau+\frac{b}{n}, n\omega). \tag{4.15}$$

Now, we can write the Hecke operators T_n in terms of U and V as

$$T_n = \sum_{d|N} V_a U_d \tag{4.16}$$

where $a := N/d$, as usual. Note that it follows immediately from the above relations on U,V that if $(n_1,n_2)=1$ then

$$T_{n_1}T_{n_2} = T_{n_2}T_{n_1} = T_{n_1 n_2}. \tag{4.17}$$

Now, let p be a prime number. A simple computation using (4.14) shows that

$$T_p T_{p^r} = T_{p^{r+1}} + \frac{1}{p} T_{p^{r-1}} W_p. \tag{4.18}$$

These relations are elegantly summarized in the formula

$$\sum_{r=0}^{\infty} T_{p^r} X^r = \frac{1}{1 - X T_p + X^2 \frac{1}{p} W_p} \tag{4.19}$$

Now, since $T_p = U_p + V_p$ we have

$$1 - X T_p + X^2 \frac{1}{p} W_p = (1 - X U_p)(1 - X V_p) \tag{4.20}$$

Thus, putting $X = p^{-s}$, we may write the elegant formula summarizing the Hecke operators in terms of U_p, V_p:

$$\sum N^{-s} T_N = \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s} V_p} \right) \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s} U_p} \right) \tag{4.21}$$

Finally, we would like to mention a physical interpretation of the various operators we have just discussed. The operators U_n and V_n have the physical interpretation of "creating" the long strings and short strings of the symmetric product, respectively. Thus, (4.16) has the interpretation that U_d "creates" a long string background and V_a then "adds" short string excitations to it. If we take an *ensemble* of theories $\mathrm{Sym}^N(\mathcal{S})$ for all N (as in Matrix theory) then the summation on the LHS of (4.21) might have a physical meaning. Then the Euler product formula might be very interesting from a physical point of view.

4.3. Multiple symmetric products

An interesting variation on the symmetric product construction is that of multiple symmetric products $\mathrm{Sym}^N\mathrm{Sym}^M(\mathcal{C}_0)$. This may be thought of as an orbifold of $\mathcal{C}_0^{NM}$ by the *wreath product* $S_N \wr S_M$, which is a proper subgroup of S_{NM}.

The wreath product is best thought of by thinking of a set of NM objects partitioned into N sets of M objects each, e.g. points X^{ia} with $1 \leq i \leq M$ and $1 \leq a \leq N$. We are allowed to permute the i's for a fixed $a = a_0$, holding X^{ia} fixed for $a \neq a_0$. We are also allowed to permute $X^{ia} \to X^{i\sigma(a)}$.

The amusing point about this construction is that, for N, M relatively prime, the index I_1 is the *same* as that for the symmetric product $\mathrm{Sym}^{NM}(\mathcal{C}_0)$. However, when $(N, M) > 1$ the index is different, because of (4.18). Some examples of this are given in Appendix G. Thus $\mathrm{Sym}^{NM}(\mathcal{C}_0)$ and $\mathrm{Sym}^N\mathrm{Sym}^M(\mathcal{C}_0)$ are distinct theories. Indeed we can keep iterating to produce a large variety of theories.

These theories offer an interesting cautionary tale in thinking about the AdS/CFT correspondence, because they show that there are *many* different CFT's with $\mathcal{A}_\gamma$ symmetry and small gap above the ground state in the Ramond sector, and hence the qualitative spectra of black holes cannot be used to deduce that, say, the holographic dual of $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$ is $\mathrm{Sym}^N(\mathcal{S})$.

5. The index for $\mathrm{Sym}^N(\mathcal{S})$

In this section we present a computation for a family of theories with $\mathcal{A}_\gamma$ symmetry. We will consider the index for the theory $\mathcal{C} = \mathrm{Sym}^N(\mathcal{S})$. This theory has $\mathcal{A}_\gamma$ symmetry with $k^+ = k^- = N$, hence $k = 2N$, $c = 3N$.

The main motivation for the computation we are going to present is the search for a holographic dual for string theory on $AdS_3 \times \mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^1$. It was proposed in [27,17] that under certain conditions on the fluxes of RR and NSNS tensor fields, the holographic dual is on the moduli space of the above theory. See [18] for more details, and for the applications of the computation of the present section.

5.1. Summary of results for the index

The formula for $I_2(\mathcal{C})$ is rather complex. To simplify matters, we assume that N is prime and we restrict attention to the charge zero sector. (Generalizations for nonprime N and nonzero charge are in sections 5.6, 5.8 below.) The result for the left-right index is

$$I_2^0(\mathcal{C}) = (N+1)|\Theta^-_{N,k}|^2 + \sum_{\substack{1\le\mu\le 2N-1\\ \mu=1 \mod 2}} \left|\Theta^-_{\mu,k} + \Theta^-_{2N-\mu,k}\right|^2 \tag{5.1}$$

Here and below the conjugation operation implied in $|\Theta|^2$ takes $\omega_\pm \to \tilde{\omega}_\pm$ and acts as complex conjugation.

As for the index $I_1(\mathcal{C})$, Ξ_μ defined in (3.11) has the form $\Xi^{ss}_\mu + \Xi^{ls}_\mu$, corresponding to the contribution of short strings and long strings. We find that

$$\Xi^{ss}_\mu = \delta_{\mu,N} \sum_{\ell^-=1/2}^{N/2} (-1)^{2\ell^- - 1} \mathrm{SCh}\Big(\frac{N+1}{2} - \ell^-, \ell^-\Big) + \cdots \tag{5.2}$$

where here and below $+\cdots$ indicates long representations.

Moreover we have

$$\Xi^{ls}_\mu = \mathrm{SCh}\Big(\frac{\mu+1}{4}, \frac{\mu+1}{4}\Big) + \mathrm{SCh}\Big(\frac{N+1}{2} - \frac{\mu+1}{4}, \frac{N+1}{2} - \frac{\mu+1}{4}\Big) + \cdots \tag{5.3}$$

for $1 \le \mu \le 2N-1$, $\mu = 1 \mod 4$, and $\mu \neq N$, while

$$\Xi^{ls}_\mu = -\mathrm{SCh}\Big(\frac{\mu-1}{4}, \frac{\mu-1}{4}\Big) - \mathrm{SCh}\Big(\frac{N+1}{2} - \frac{\mu-1}{4}, \frac{N+1}{2} - \frac{\mu-1}{4}\Big) + \cdots \tag{5.4}$$

for $3 \le \mu \le (2N-3)$, $\mu = 3 \mod 4$, and $\mu \neq N$. If $\mu = N$ we have

$$\Xi^{ls}_{\mu=N} = (-1)^{\frac{1}{2}N(N-1)} \mathrm{SCh}\Big(\frac{N+1}{4}, \frac{N+1}{4}\Big) + \cdots \tag{5.5}$$

The simplest RR spectrum consistent with these results is

$$\begin{gathered} \bigoplus_{\ell^-=1/2}^{N/2} \left|\Big(\frac{N+1}{2} - \ell^-, \ell^-\Big)\right|^2 \\ \bigoplus_{\ell=1/2}^{(N-1)/4} \left|(\ell,\ell) + \Big(\frac{N+1}{2} - \ell, \frac{N+1}{2} - \ell\Big)\right|^2 \oplus \left|\Big(\frac{N+1}{4}, \frac{N+1}{4}\Big)\right|^2 \end{gathered} \tag{5.6}$$

where the first line comes from the short string and the second from the long string contribution. The short string states have $h = \frac{N}{4}$ for all states, and the gap to the next excited state is order 1. The long string states $(a = 1, d = N)$ have

$$h = \frac{N}{8} + \frac{(4\ell - 1)^2}{8N} \tag{5.7}$$

and have small gaps $\sim 1/N$ to the first excited state.

While this is the simplest spectrum, it is not the unique spectrum consistent with the index. The true spectrum might differ by a virtual representation of index zero. These can always be expressed in terms of massive representations and are expected to disappear upon turning on a generic perturbation – barring some extra symmetry protecting states.

Applying spectral flow to the representation (5.6) gives:

$$\begin{aligned} &\bigoplus_{j=0}^{(N-1)/2} |(j,j)_{NS}|^2 \\ &\bigoplus_{j=0}^{(N-3)/4} \left|(\frac{N-1}{2} - j, j)_{NS} + (j, \frac{N-1}{2} - j)_{NS}\right|^2 \oplus \left|\Big(\frac{N-1}{4}, \frac{N-1}{4}\Big)_{NS}\right|^2 \end{aligned} \tag{5.8}$$

where the first line is from the short string contribution $a = N, d = 1$ and the second from the long string contribution $a = 1, d = N$. The short string states have $h = j$ while the long string states have:

$$h = \frac{N-1}{4} + \frac{(N-1-4j)^2}{8N}. \tag{5.9}$$

In subsections 5.3 - 5.5 we give the proofs of these results.

5.2. Interpreting states contributing to the index

Before presenting the technical computation we give a direct construction of a set of states that accounts for the nonvanishing index $I_2(\mathcal{C})$.

Short string states $(a = N, d = 1)$: Here we are looking for (bps, bps) states in $\mathrm{Sym}^N(\mathcal{H})$ where $\mathcal{H}$ is the Hilbert space of $\mathcal{S}$. Again we must stress that this is a graded tensor product. The states are most simply described in the NS sector. We consider states built out of the fermion operators $\psi^a_{-1/2}(i)$ (where (i) refers to the factor in the symmetric product). We may then introduce the operator

$$\mathcal{O}^{ab} = \sum_i \psi^a_{-\frac{1}{2}}(i)\tilde{\psi}^b_{-\frac{1}{2}}(i) \tag{5.10}$$

Amongst the states:

$$\mathcal{O}^{a_1 b_1} \cdots \mathcal{O}^{a_n b_n} |0\rangle_{NS} \tag{5.11}$$

there is a subspace of states transforming as $SU(2)^4$ multiplets $(\frac{n}{2}, \frac{n}{2}; \frac{n}{2}, \frac{n}{2})$, and having $h = n/2$. They are therefore BPS multiplets. Moreover, for $N > 1$ they are *not* descendents in the BPS multiplet $|(0,0)|^2$ since the descendent fermion operator is $Q^a_{-\frac{1}{2}} = \sum_{i=1}^N \psi^a_{-\frac{1}{2}}(i)$. Finally, the tower cuts off at $n = N-1$ by the Fermi statistics of $\psi^a_{-1/2}(i)$, i.e. for $n = N$ the state (5.11) is a descendent $|Q^{a_1}_{-\frac{1}{2}} \cdots Q^{a_N}_{-\frac{1}{2}}|^2 |0\rangle$.

Long string states: $(a = 1, d = N)$. Here we may use the inheritance principle of (4.9). If we identify the states in the $\mathcal{S}$ theory which contribute to the index for $\mathcal{S}$, then we automatically find corresponding states in the long-string sector contributing to the index. We will see below that the index for $\mathcal{S}$ comes from states with $h - c/24 = \frac{1}{2}m(m+1)$, $m \geq 0$ with spins of the form:

$$\psi^{+,+}_{-1} \psi^{+,+}_{-2} \cdots \psi^{+,+}_{-m} |\Omega_\pm\rangle \tag{5.12}$$

where all the $SU(2) \times SU(2)$ spins are aligned, we have used bispinor notation for the fermions and $\Omega_\pm$ are the two highest weight states for the Ramond ground state of $\mathcal{S}$. Now, applying the inheritance principle we find a collection of multiplets contributing $|(\ell, \ell)|^2$ as in the second line of (5.6). By suitably flipping spins one can obtain the off-diagonal representations in the same way (*c.f.* section 5.3 of [18]).

Let us comment on BPS states which *do not* contribute to the index. There is an algorithm by which one can, in principle, determine the exact spectrum (not just BPS) of the theory $\mathcal{C}$ using the algebra of level k theta functions. The reason is that Z_0 for $\mathcal{S}$ can be expressed in terms of even level 1 theta functions $\Theta^+_{\mu,1}(\omega_+, \tau)$ and $\Theta^+_{\mu,1}(\omega_-, \tau)$. The Hecke transform T_M maps these to even level M theta functions. The even theta functions form an algebra, and the partition function of $\mathcal{C}$ can be expressed as a *linear* combination of even level N theta functions of $\omega_\pm, \tilde{\omega}_\pm$ (with modular forms as coefficients). Finally, the space of affine $SU(2)$ characters $\chi^{(k)}_\mu$ is spanned by even level k theta functions. Putting these facts together gives an algorithm for determining the exact representation content of the theory $\mathrm{Sym}^N(\mathcal{S})$. We have not carried this out partly because the expressions are complicated, and partly because it is only the index which is expected to be an invariant on the moduli space of the $\mathrm{Sym}^N(\mathcal{S})$ theory. Nevertheless, it is easy to identify some interesting BPS states which come in cancelling pairs, and this we now describe.

One set of interesting BPS states comes in twisted sectors associated to the conjugacy class $(n)(1)^{N-n}$ for $0 < n < N$. These states live in the summand

$$\mathcal{H}^{(n)} \otimes \mathrm{Sym}^{N-n}(\mathcal{H}) \tag{5.13}$$

of (4.1). In the RR sector the character will have a second order zero at $z_+ = z_-$ since it is the product of two characters for $\mathcal{A}_\gamma$ at $(k^+, k^-) = (n, n)$ and at $(N-n, N-n)$ respectively. Thus, we can expand in terms of massive characters. Nevertheless we should examine these sectors more closely since the massive characters are at threshhold. In the NS sector of $\mathcal{H}^{(n)}$, with n odd, one can construct explicitly [18] BPS states with

$$h = \ell^+ = \ell^- = \frac{(n-1)}{4} \tag{5.14}$$

These give BPS states when combined with certain other states in $\mathrm{Sym}^{N-n}(\mathcal{H})$. One way to do this is by tensoring the BPS state (5.14) with any of the four states:

$$\begin{aligned}
&|0\rangle_{\mathrm{left}} \otimes |0\rangle_{\mathrm{right}} \\
&\Big(\sum_{i=1}^{(N-n)} Q^a_{-1/2}(i)\Big)|0\rangle_{\mathrm{left}} \otimes |0\rangle_{\mathrm{right}} \\
&|0\rangle_{\mathrm{left}} \otimes \Big(\sum_{i=1}^{(N-n)} \tilde{Q}^a_{-1/2}(i)|0\rangle_{\mathrm{right}}\Big) \\
&\Big(\sum_{i=1}^{(N-n)} Q^a_{-1/2}(i)\Big)|0\rangle_{\mathrm{left}} \otimes \Big(\sum_{i=1}^{(N-n)} \tilde{Q}^a_{-1/2}(i)|0\rangle_{\mathrm{right}}\Big)
\end{aligned} \tag{5.15}$$

If the spin a is aligned with that of (5.14) then the resulting state will be BPS, but for $n < N$ it is not a descendent. Alternatively, we can consider

$$\begin{aligned}
&\Phi \otimes |0\rangle_{N-n} \\
&Q^{(n)}_{-\frac{1}{2}} \Phi \otimes |0\rangle_{N-n} \\
&\tilde{Q}^{(n)}_{-\frac{1}{2}} \Phi \otimes |0\rangle_{N-n} \\
&Q^{(n)}_{-\frac{1}{2}} \tilde{Q}^{(n)}_{-\frac{1}{2}} \Phi \otimes |0\rangle_{N-n}
\end{aligned} \tag{5.16}$$

These also form a cancelling quartet. In the terminology of the AdS/CFT correspondence (5.16) are "single particle states" while (5.15) are "multiparticle states."

A second way in which one can make new BPS states from the states (5.14) is by tensoring (in the NS sector) with the short string BPS states in $\mathrm{Sym}^{N-n}(\mathcal{H})$, which are

analogous to the states (5.11) constructed above. The result is a set of BPS representations with

$$
(h,\ell^+,\ell^-) = (\frac{n-1}{4}+\frac{s}{2},\frac{n-1}{4}+\frac{s}{2},\frac{n-1}{4}+\frac{s}{2}) \qquad 0\le s\le N-n-1 \tag{5.17}
$$

on both left and right. Under spectral flow to the RR sector these states contribute to the index $(-1)^{(n-1)/2+s}\Theta_{N,k}$. For N odd the sum over s cancels out. For N even there are other states cancelling the index. Further examples of BPS states not contributing to the index can be found in [17] eq. 5.10.

5.3. Preliminary results on the basic $c=3$ theory $\mathcal{S}$

Let us assume that the boson φ in the $\mathcal{S}$ theory has radius R. Then the RR-sector supercharacter before GSO projection is

$$
Z_0 = \Big|\mathrm{SCh}^{\mathcal{S},R}(\tau,\omega_+,\omega_-)\Big|^2 Z_\Gamma \tag{5.18}
$$

where

$$
Z_\Gamma = \sum_{\Gamma^{1,1}} q^{\frac{1}{2}p_L^2}\overline{q}^{\frac{1}{2}p_R^2} \tag{5.19}
$$

is a standard Siegel-Narain theta function for radius R.

The supercharacter is given by

$$
\begin{aligned}
&\mathrm{SCh}^{\mathcal{S},R}(\tau,\omega_+,\omega_-) = q^{1/8}(z_+ + z_+^{-1} - z_- - z_-^{-1})\prod(1-q^n)^{-1}\\
&\prod_{n>0}(1-z_+z_-q^n)(1-z_+^{-1}z_-^{-1}q^n)(1-z_+^{-1}z_-q^n)(1-z_+z_-^{-1}q^n)
\end{aligned} \tag{5.20}
$$

The q expansion of (5.20) defines a series of representations R_n of $SU(2)\times SU(2)$ via:

$$
\mathrm{SCh}^{\mathcal{S},R} := q^{1/8}\sum_{n=0}^{\infty} q^n \chi_{R_n}(z_+,z_-) \tag{5.21}
$$

where R_n is a supercharacter of a (reducible) representation of $SU(2)\times SU(2)$. It is useful to introduce the $SU(2)$ characters of the irreducible representations of spin ℓ:

$$
\chi_\ell(z) := \frac{z^{2\ell+1}-z^{-2\ell-1}}{z-z^{-1}} = z^{-2\ell}+z^{-2\ell+2}+\cdots+z^{+2\ell} \tag{5.22}
$$

In particular, we define $u=\chi_{1/2}(z_+)$ and $v=\chi_{1/2}(z_-)$. Clearly, (5.20) vanishes at $z_+=z_-$ and therefore $\chi_{R_n}=(u-v)p_n(u,v)$ where $p_n(u,v)$ is a polynomial in u,v.

In our expansion of I_1 in terms of $\mathcal{A}_\gamma$ characters we will need the following crucial property: $\chi_{R_n}(z_+,z_-)$ has a first order zero iff n is a triangular number, that is, iff $n = \frac{1}{2}m(m+1)$ for some integer m. Moreover, for $n=\frac{1}{2}m(m+1)$ we have

$$p_{\frac{1}{2}m(m+1)}(z_+,z_-) = (-1)^m\Big(\chi_{m/2}(z_+)\chi_{m/2}(z_-) - \chi_{m/2-1/2}(z_+)\chi_{m/2-1/2}(z_-)\Big) + (u-v)q_m(u,v) \tag{5.23}$$

where $q_m(u,v)$ is a symmetric polynomial in u,v.

To prove this note that by applying (3.3) to $\mathrm{SCh}^{\mathcal{S},R}$, using the infinite product and then the infinite sum representation of ϑ_1 we find:

$$-z_+\frac{d}{dz_-}\Big|_{z_-=z_+=z}\mathrm{SCh}^{\mathcal{S},R} = q^{1/8}\sum_{m=0}^{\infty} q^{\frac{1}{2}m(m+1)}(-1)^m(z^{2m+1}-z^{-2m-1}) \tag{5.24}$$

Now note that

$$-z_+\frac{d}{dz_-}\Big|_{z_-=z_+=z}\big[(u-v)p_n(u,v)\big] = (z-z^{-1})p_n(u,u) \tag{5.25}$$

It thus follows from (5.24) that $p_n(u,u)=0$ unless $n=\frac{1}{2}m(m+1)$ is a triangular number. Moreover, in the case that n is triangular (5.24) and (5.25) implies

$$p_n(u,u) = (-1)^m\frac{(z^{2m+1}-z^{-2m-1})}{z-z^{-1}} \tag{5.26}$$

Now note that putting $z_+=z_-=z$ in (5.23) and using the Clebsch-Gordon decomposition we get $(-1)^m\chi_m(z)$, in agreement with (5.26).

We conclude this subsection with an important remark. The argument given above for (5.23) only establishes the existence of $q_m(u,v)$ as a *virtual character* of $SU(2)\times SU(2)$. Indeed, define

$$\varphi(j_1,j_2) = \chi_{j_1}(z_+)\chi_{j_2}(z_-) - \chi_{j_1-1/2}(z_+)\chi_{j_2-1/2}(z_-). \tag{5.27}$$

Observe that

$$\varphi(j_1,j_2) = \varphi(j_1-1/2,j_2+1/2) + (u-v)\chi_{j_1-1/2}(z_+)\chi_{j_2(z_-)} \tag{5.28}$$

This identity indeed establishes the leading term in the q-expansion of (3.7). Therefore, in general $\varphi(j_1,j_2)=\varphi(j_1-j_3,j_2+j_3)$ mod $(u-v)$. The property that distinguishes the choice in (5.23) is that this is the unique choice so that $\varphi(j_1,j_2)$ has a positive coefficient and $q_m(u,v)$ is a *positive integer* combination of supercharacters. This follows from an infinite product expansion for the character. Note also that this is the unique choice which is symmetric under $u\leftrightarrow v$.

5.4. Computation of the index $I_2(\mathcal{C})$

In order to compute the indices we need the following basic technical result. The term associated with a, b, d in the sum (4.4) for T_N is computed using

$$\begin{aligned} -z_+ \frac{d}{dz_-}\Big|_{z_-=z_+} &\mathrm{SCh}^{\mathcal{S},R}\Big(\frac{a\tau+b}{d}, a\omega_+, a\omega_-\Big) = \\ &= a(z_+^a - z_+^{-a})\tilde{q}^{1/8}\prod_{n>0}(1-\tilde{q}^n)(1-z_+^{2a}\tilde{q}^n)(1-z_+^{-2a}\tilde{q}^n) \\ &= -ia\vartheta_1\Big(2a\omega, \frac{a\tau+b}{d}\Big) = a\Theta^-_{1,2}\Big(a\omega, \frac{a\tau+b}{d}\Big) \\ &= a\sum_{n_0=0}^{d-1} e^{4\pi i \frac{b}{d}(n_0+\frac{1}{4})^2}\Theta^-_{a(4n_0+1),2N}(\omega,\tau) \end{aligned} \tag{5.29}$$

where $N = ad$, and $z = z_+ = e^{2\pi i\omega}$, $\tilde{q} = \exp[2\pi i\frac{a\tau+b}{d}]$ and we used the identities (D.7) and (D.8). Applying (5.29) to both the left-movers and the right-movers we get

$$\begin{aligned} I_2(\mathrm{Sym}^N(\mathcal{S})) = \sum_{ad=N}\sum_{n_0,m_0=0}^{d-1} &a\Theta^-_{a(4n_0+1),k}(\omega,\tau)\overline{\Theta^-_{a(4m_0+1),k}(\tilde{\omega},\tilde{\tau})}\cdot \\ &\cdot\frac{1}{d}\sum_{b=0}^{d-1} e^{2\pi i\frac{b}{d}(n_0-m_0)(2n_0+2m_0+1)}Z_\Gamma(\frac{a\tau+b}{d}) \end{aligned} \tag{5.30}$$

where Z_Γ is defined in (5.19).

The sum on b becomes a projection operator, and depends on the charge sector of the theory. Here we focus on the charge zero sector with $u = \tilde{u} = 0$. In this case the index simplifies to

$$I_2^0 = \sum_{ad=N}\sum_{n_0,m_0=0}^{d-1} a\Pi^{(d)}(n_0,m_0)\Theta^-_{a(4n_0+1),k}(\omega,\tau)\overline{\Theta^-_{a(4m_0+1),k}(\tilde{\omega},\tilde{\tau})} \tag{5.31}$$

where we have introduced the projector $\Pi^{(d)}(n_0,m_0)$ onto

$$(n_0-m_0)(2n_0+2m_0+1) = 0 \bmod d \tag{5.32}$$

Now we will assume N prime. In this case there are only two contributions: $a = N, d = 1$ and $a = 1, d = N$ in the sum over coverings. The "short string" contribution $a = N, d = 1$ contributes $N|\Theta^-_{N,k}|^2$. The "long string" contribution $a = 1, d = N$ is:

$$\sum_{n=0}^{N-1}|\Theta^-_{4n+1,k}|^2 + \sum_{n=0,\mu\neq N}^{N-1}\Theta^-_{4n+1,k}\overline{\Theta^-_{2N-(4n+1),k}} \tag{5.33}$$

Adding the contributions and using $\Theta^-_{\mu,k} = -\Theta^-_{2k-\mu,k}$ we arrive at (5.1).

5.5. Computation of I_1

Applying (5.29) only on the left-movers produces the left-index:

$$\begin{aligned} I_1(\mathrm{Sym}^N(\mathcal{S})) = & \sum_{ad=N}\sum_{n_0=0}^{d-1}\Theta^-_{a(4n_0+1),k}(\omega,\tau) \\ & \frac{1}{d}\sum_{b=0}^{d-1} e^{4\pi i\frac{b}{d}(n_0+\frac{1}{4})^2}\left[\overline{\mathrm{SCh}^{\mathcal{S},R}}\sum q^{\frac{1}{2}p_L^2}\overline{q}^{\frac{1}{2}p_R^2}\right]\Big(\frac{a\tau+b}{d};\frac{a\overline{\tau}+b}{d},a\tilde{\omega}_+,a\tilde{\omega}_-\Big)\end{aligned} \tag{5.34}$$

We can now extract the contribution Ξ_μ defined in (3.11). Again we will restrict attention to the charge zero sector and we will assume N is prime.

5.5.1.*Evaluation of the $a=N, d=1$ term*

We take the derivative in the index on the left. Therefore we have to expand

$$\mathrm{SCh}^{\mathcal{S},R}(N\tau,N\omega_+,N\omega_-) \tag{5.35}$$

in terms of $\mathcal{A}_\gamma$ characters for $k=2N$. Note that applying (5.29) to (5.35) leads to

$$N\Theta^-_{N,k}(\omega,\tau) \tag{5.36}$$

Thus, we must find at least N representations with $\ell^+ + \ell^- = (N+1)/2$.

The $\mathcal{A}_\gamma$ highest weight states turn out to contribute only to the leading term in the q-expansion of (5.35). The spins of the BPS representations can then be extracted by examining the $SU(2)\times SU(2)$ character in this term and matching to the leading term in the q-expansion of the characters of BPS representations of $\mathcal{A}_\gamma$. These leading terms are summarized in appendix A.

The leading term in the q expansion of (5.35) is

$$q^{N/8}\Big((\chi_{N/2}(z_+)-\chi_{(N-2)/2}(z_+))-(\chi_{N/2}(z_-)-\chi_{(N-2)/2}(z_-)\Big)+\cdots \tag{5.37}$$

In order to match this to the leading terms in the characters of $\mathcal{A}_\gamma$ we need to factor out $u-v$ from the character in (5.37). This is conveniently done using the relation of $SU(2)$ characters to Chebyshev polynomials U_n. This leads to the key identity:

$$\chi_{1/2}(z_+^N)-\chi_{1/2}(z_-^N) = (u-v)\sum_{a+b=N-1}\Big[\chi_{a/2}(z_+)\chi_{b/2}(z_-)-\chi_{(a-1)/2}(z_+)\chi_{(b-1)/2}(z_-)\Big] \tag{5.38}$$

where the sum is on integers $a,b\geq 0$, and we are using the identity $\chi_{-1/2}(z)=0$.

After a little algebra one arrives at

$$\Xi_N^{ss} = \mathrm{SCh}^{\mathcal{S},R}\Big(N\tau,z_+^N,z_-^N\Big) = \sum_{\ell^-=1/2}^{N/2}(-1)^{2\ell^--1}\mathrm{SCh}(\frac{N+1}{2}-\ell^-,\ell^-)+\cdots \tag{5.39}$$

The higher terms all involve massive characters with signs.

Example. $N = 5,\ a = 5,\ d = 1$

$$
\begin{aligned}
\Xi^{sg}_{\mu=5} = &\, \mathrm{SCh}(5/2,1/2) + \mathrm{SCh}(1/2,5/2) + \mathrm{SCh}(3/2,3/2) - \\
& - \mathrm{SCh}(2,1) - \mathrm{SCh}(1,2) + \\
& + \mathrm{SCh_m}(13/8,5/2,2/2) + \mathrm{SCh_m}(13/8,2/2,5/2) + \\
& + \mathrm{SCh_m}(21/8,5/2,2/2) + \mathrm{SCh_m}(21/8,2/2,5/2) - \\
& - 2\mathrm{SCh_m}(21/8,4/2,3/2) - 2\mathrm{SCh_m}(21/8,3/2,4/2) + \\
& + \mathrm{SCh_m}(29/8,5/2,2/2) + \mathrm{SCh_m}(29/8,2/2,5/2) + \\
& + 3\mathrm{SCh_m}(29/8,4/2,3/2) + 3\mathrm{SCh_m}(29/8,3/2,4/2) - \\
& - 3\mathrm{SCh_m}(37/8,5/2,2/2) - 3\mathrm{SCh_m}(37/8,2/2,5/2) - \\
& - 2\mathrm{SCh_m}(37/8,4/2,3/2) - 2\mathrm{SCh_m}(37/8,3/2,4/2) - \\
& - 2\mathrm{SCh_m}(45/8,5/2,2/2) - 2\mathrm{SCh_m}(45/8,2/2,5/2) + \\
& + \mathrm{SCh_m}(45/8,4/2,3/2) + \mathrm{SCh_m}(45/8,3/2,4/2) + \\
& + \ldots
\end{aligned}
\tag{5.40}
$$

Here, we introduced a shorthand notation, $\mathrm{SCh_m}(h,\ell^+,\ell^-) := \mathrm{SCh_{massive}}(h+\frac{c}{24},\ell^+,\ell^-)$. Notice, that the massless supercharacters in the first two lines agree with the decomposition (5.39). However, the multiplicities of massive representations do not have a definite sign.

Comparing (5.39) with the expansion (3.12) shows that the degeneracies $n^{ss}(\ell^+,\ell^-;\tilde{\ell}^+,\tilde{\ell}^-)$ vanish unless $\mu = \tilde{\mu} = N$, consistent with the index I_2. Moreover, for $\tilde{\mu} = N$ we have

$$
\sum_{\ell^+ + \ell^- = (N+1)/2} (-1)^{2\ell^- + 1} n^{ss}(\ell^+,\ell^-;\tilde{\ell}^+,\tilde{\ell}^-) = (-1)^{2\tilde{\ell}^- + 1} \tag{5.41}
$$

for each representation with $\tilde{\ell}^+ = (N+1)/2 - \tilde{\ell}^-$, $\frac{1}{2} \le \tilde{\ell}^- \le N/2$. The simplest spectrum consistent with the index is

$$
\bigoplus (\ell^+,\ell^-)_S \otimes \overline{(\ell^+,\ell^-)_S} \tag{5.42}
$$

where the sum is over all spins, integer and half-integer with

$$
\ell^+ + \ell^- = (N+1)/2, \qquad \ell^\pm \ge 1/2 \tag{5.43}
$$

Again, we caution that (5.42) is only the simplest spectrum consistent with the index I_1.

5.5.2. *Evaluation of the $a = 1, d = N$ term*

Now the "long string" contribution to Ξ_μ is

$$\Xi^{ls}_\mu := \frac{1}{N}\sum_{b=0}^{N-1} e^{-4\pi i \frac{b}{N}\left(n_0+\frac{1}{4}\right)^2} \mathrm{SCh}^{\mathcal{S},R}\Big(\frac{\tau+b}{N}, \tilde\omega_+, \tilde\omega_-\Big) \tag{5.44}$$

Here $\mu = 4n_0+1$, $n_0 = 0,\dots,N-1$. While Ξ_μ was defined in (3.12) only for $1 \le \mu \le k-1$ we extend the range of definition of μ in the same was as for $\Theta^-_{\mu,k}$. Thus, $\Xi_{\mu+4N} = \Xi_\mu$ and $\Xi_{4N-\mu} = -\Xi_\mu$.

Now, we would like to decompose (5.44) in terms of characters of $\mathcal{A}_\gamma$ at level $(k^+,k^-) = (N,N)$ and in particular we would like to isolate the contribution of the massless characters. To this end we use the q-expansion (5.21). The projection from the sum over b tells us that only the terms with

$$n = 2n_0^2 + n_0 \bmod N \tag{5.45}$$

survive in the sum (5.44) and hence

$$\Xi^{ls}_\mu = \sum_{n\ge 0, n=n_0(2n_0+1) \bmod N} e^{2\pi i \frac{\tau}{N}(n+1/8)} \chi_{R_n}(z_+,z_-) \tag{5.46}$$

(We have used the definition (5.21).) Now, to extract the BPS representations we use the leading q expansion of BPS characters given in appendix A. Accordingly, we look for n such that χ_{R_n} has a first order zero at $z_+ = z_-$. As discussed in sec. 4.3 this happens iff n is a triangular number $n = \frac{1}{2}m(m+1)$, in which case, by (5.23) the $SU(2)\times SU(2)$ character is that of the leading term in the q-expansion of the character of the short $(\frac{1}{2}m+\frac{1}{2}, \frac{1}{2}m+\frac{1}{2})$ representation of $\mathcal{A}_\gamma$. Thus, a necessary condition for the n^{th} term in (5.46) to correspond to the leading term in the expansion of a BPS character is that $n = \frac{1}{2}m(m+1)$, in which case the spins must be $(\frac{1}{2}m+\frac{1}{2}, \frac{1}{2}m+\frac{1}{2})$. Note that such quantum numbers indeed saturate the BPS bound:

$$k(L_0 - c/24) = 2N\frac{1}{N}(n+\frac{1}{8}) = (m+1/2)^2 = (\ell^+ + \ell^- - 1/2)^2 \tag{5.47}$$

While there are infinitely many n which satisfy the above criteria only finitely many correspond to $\mathcal{A}_\gamma$ highest weight states since by the unitarity constraints we know that $0 \le m \le (N-1)$. Moreover, by the projection (5.45) we have

$$(m-2n_0)(m+2n_0+1) = 0 \bmod N \tag{5.48}$$

We now solve these constraints: For $0 \le n_0 \le (N-1)/2$ the only solutions are $m = 2n_0$ and $m = (N-1) - 2n_0$. For $(N+1)/2 \le n_0 \le (N-1)$ the only solutions are $m = 2n_0 - N$ and $m = (N-1) - (2n_0 - N)$. This shows which BPS representations *might* occur. On the other hand, the index I_2 shows that these representations, at least, must occur.

Thus, we conclude that for $0 \le n_0 \le (N-1)/2$, and $n_0 \ne (N-1)/4$ we have

$$\Xi^{ls}_{\mu} = \mathrm{SCh}(n_0 + 1/2, n_0 + 1/2) + \mathrm{SCh}\Big(\frac{N-1}{2} - n_0 + 1/2, \frac{N-1}{2} - n_0 + 1/2\Big) + \cdots \quad (5.49)$$

Note that this corresponds to the range $1 \le \mu \le 2N-1$, $\mu = 1 \bmod 4$, and $\mu \ne N$, and can be expressed as:

$$\Xi^{ls}_{\mu} = \mathrm{SCh}\Big(\frac{\mu+1}{4}, \frac{\mu+1}{4}\Big) + \mathrm{SCh}\Big(\frac{N+1}{2} - \frac{\mu+1}{4}, \frac{N+1}{2} - \frac{\mu+1}{4}\Big) + \cdots \quad (5.50)$$

For $(N+1)/2 \le n_0 \le (N-1)$, and $n_0 \ne (3N-1)/4$ we have

$$\Xi^{ls}_{\mu} = \mathrm{SCh}\Big(n_0 - \frac{N}{2} + 1/2, n_0 - \frac{N}{2} + 1/2\Big) + \mathrm{SCh}(N - n_0, N - n_0) + \cdots \quad (5.51)$$

Note this corresponds to the range $2N+3 \le \mu \le 4N-3$, $\mu = 1 \bmod 4$, $\mu \ne 3N$. It is more convenient to work with μ in the original range $1 \le \mu \le 2N-1$, so we use $\Xi_{\mu} = -\Xi_{4N-\mu}$ to say

$$\Xi^{ls}_{\mu} = -\mathrm{SCh}\Big(\frac{\mu-1}{4}, \frac{\mu-1}{4}\Big) - \mathrm{SCh}\Big(\frac{N+1}{2} - \frac{\mu-1}{4}, \frac{N+1}{2} - \frac{\mu-1}{4}\Big) + \cdots \quad (5.52)$$

where now $3 \le \mu \le (2N-3)$, $\mu = 3 \bmod 4$, and $\mu \ne N$. Finally, if $\mu = N$ we have

$$\Xi^{ls}_{\mu=N} = \epsilon \mathrm{SCh}\Big(\frac{N+1}{4}, \frac{N+1}{4}\Big) + \cdots \quad (5.53)$$

where $\epsilon = (-1)^{\frac{1}{2}N(N-1)}$.

It thus follows that there are nonzero degeneracies for all representations (ℓ, ℓ), $1/2 \le \ell \le N/2$. The simplest spectrum consistent with the index is

$$\bigoplus_{\ell=1/2}^{(N-1)/4} \Big|(\ell,\ell) + \Big(\frac{N+1}{2} - \ell, \frac{N+1}{2} - \ell\Big)\Big|^2 \oplus \Big|\Big(\frac{N+1}{4}, \frac{N+1}{4}\Big)\Big|^2 \quad (5.54)$$

where all spins, integer and half-integer appear.

Again, note that the arguments above only establish that the terms $+\cdots$ above are linear combinations of massive supercharacters with signs. However, because of (3.7) this leaves ambiguous the short representations which appear. This ambiguity may be removed

as follows. Note, that from the index I_2 the spectrum of massless representations is diagonal up to reflection of spins. This property can be used to constrain the multiplicities of both massless and massive supercharacters that appear in the decomposition of Ξ_μ^{ls}. Indeed, from (3.12) we find that the multiplicities of massless supercharacters, $\mathrm{SCh}(\tilde{\ell}^+, \tilde{\ell}^-)$, are given by $(-1)^{\frac{\mu+3}{2}} n(\ell, \ell; \tilde{\ell}, \tilde{\ell})\ \delta_{\mu,4\ell-1}$, whereas the multiplicities of massive supercharacters, $\mathrm{SCh}(\tilde{\rho}_{\mathrm{m}})$, are given by $(-1)^{\frac{\mu+3}{2}} N(\ell, \ell; \tilde{\rho}_{\mathrm{m}})$. Since the degeneracies $N(\rho, \tilde{\rho})$ in (3.10) are *positive*, we conclude that (5.50) should be expanded in *positive integral* combinations of supercharacters, while (5.52) should be expanded in negative integral combinations of supercharacters. We verified this property (by computer) in a number of examples.[4]

Example. $N = 5,\ a = 1,\ d = 5$

$$
\begin{aligned}
\Xi^{ls}_{\mu=9} = {} & \mathrm{SCh}(1/2, 1/2) + \mathrm{SCh}(3/2, 3/2) + \\
& + 3\mathrm{SCh}_{\mathrm{m}}(41/40, 3/2, 4/2) + 3\mathrm{SCh}_{\mathrm{m}}(41/40, 4/2, 3/2) + \\
& + 6\mathrm{SCh}_{\mathrm{m}}(41/40, 3/2, 2/2) + 6\mathrm{SCh}_{\mathrm{m}}(41/40, 2/2, 3/2) + \\
& + 6\mathrm{SCh}_{\mathrm{m}}(81/40, 5/2, 4/2) + 6\mathrm{SCh}_{\mathrm{m}}(81/40, 4/2, 5/2) + \\
& + 10\mathrm{SCh}_{\mathrm{m}}(81/40, 5/2, 2/2) + 10\mathrm{SCh}_{\mathrm{m}}(81/40, 2/2, 5/2) + \\
& + 42\mathrm{SCh}_{\mathrm{m}}(81/40, 4/2, 3/2) + 42\mathrm{SCh}_{\mathrm{m}}(81/40, 3/2, 4/2) + \\
& + 61\mathrm{SCh}_{\mathrm{m}}(81/40, 3/2, 2/2) + 61\mathrm{SCh}_{\mathrm{m}}(81/40, 2/2, 3/2) + \\
& + 61\mathrm{SCh}_{\mathrm{m}}(121/40, 5/2, 4/2) + 61\mathrm{SCh}_{\mathrm{m}}(121/40, 4/2, 5/2) + \\
& + 75\mathrm{SCh}_{\mathrm{m}}(121/40, 5/2, 2/2) + 75\mathrm{SCh}_{\mathrm{m}}(121/40, 2/2, 5/2) + \\
& + 290\mathrm{SCh}_{\mathrm{m}}(121/40, 4/2, 3/2) + 290\mathrm{SCh}_{\mathrm{m}}(121/40, 3/2, 4/2) + \\
& + 348\mathrm{SCh}_{\mathrm{m}}(121/40, 3/2, 2/2) + 348\mathrm{SCh}_{\mathrm{m}}(121/40, 2/2, 3/2) + \\
& + \ldots
\end{aligned}
\tag{5.55}
$$

Notice, that massless supercharacters in the first line agree with the expansion (5.50), and that the multiplicities of massive representations are all positive and form a regular pattern.

We should stress, however, that this method allows to completely remove the ambiguity only in the $a = 1$, $d = N$ term, where Ξ_μ decomposes into massless and massive supercharacters of a definite sign.

[4] The computer code is available upon request.

5.6. Charged excitations

Let us now indicate how the results generalize to the spectrum of charged excitations in the $\mathrm{Sym}^N(\mathcal{S})$ theory.

If the radius of the boson in the $\mathcal{S}$ theory is finite, $\varphi \sim \varphi + 2\pi R$, then there will be a Narain lattice of charges. A state with momentum n and winding w contributes:

$$q^{\frac{\alpha'}{4}(n/R+wR/\alpha')^2}\overline{q}^{\frac{\alpha'}{4}(n/R-wR/\alpha')^2} \tag{5.56}$$

In units with $\alpha' = 2$ the spectrum of (p_L, p_R) is

$$(p_L, p_R) = (n/R + wR/2, n/R - wR/2) \tag{5.57}$$

with $n, w \in \mathbb{Z}$. Let us call this lattice $\Gamma(R) \subset \mathbb{R}^2$.

Given that the charges $(u_L; u_R)$ of the $\mathcal{S}$ theory form a Narain lattice $\Gamma(R)$ we would like to describe the lattice of charges

$$\Gamma_{\mathrm{Sym}^N(\mathcal{S})} := \left\{ (u_L; u_R) | \exists \psi \in \mathcal{H}(\mathrm{Sym}^N(\mathcal{S})) \text{ s.t. } U\psi = u_L\psi, \tilde{U}\psi = u_R\psi \right\} \tag{5.58}$$

of the $\mathrm{Sym}^N(\mathcal{S})$ theory, and how those charges are correlated with uncharged states. It is easy to see that the charges $(u_L; u_R)$ always lie in the same Narain lattice $\Gamma(R)$ as the parent theory $\mathcal{S}$, i.e. $\Gamma_{\mathrm{Sym}^N(\mathcal{S})} \subset \Gamma(R)$. We now discuss the charged states in more detail.

The conventionally normalized $U(1)$ current in the $\mathcal{A}_\gamma$ algebra is

$$U(z)U(w) \sim -\frac{k}{2}\frac{1}{(z-w)^2} + \cdots \tag{5.59}$$

In the $\mathcal{S}$ theory there is a scalar field φ and $U(z) = \partial\varphi$ with $k = 2$. In the $\mathrm{Sym}^N(\mathcal{S})$ theory we can form the field $\varphi_{\mathrm{Diag}} := \varphi_1 + \varphi_2 + \cdots + \varphi_N$. Notice that φ and φ_{Diag} have the *same* periodicity,

$$\varphi_{\mathrm{Diag}} \sim \varphi_{\mathrm{Diag}} + 2\pi R$$

and that

$$U := \partial\varphi_{\mathrm{Diag}}$$

is the conventionally normalized $U(1)$ current in the $\mathcal{A}_\gamma$ algebra of $\mathrm{Sym}^N(\mathcal{S})$ because $k = 2N$. It follows that in the untwisted sector the spectrum of charges is $N\Gamma(R)$. Moreover, let us note that we may bosonize the current U by introducing a conventionally normalized

effective boson Φ_{eff} by $U = \sqrt{N}\partial\Phi_{\text{eff}}$. Clearly $\Phi_{\text{eff}} = \frac{1}{\sqrt{N}}\varphi_{\text{Diag}}$ and hence it has radius $R/\sqrt{N}$. Thus, the radius of Φ_{eff} is determined by the radius of φ.

All operators in $\text{Sym}^N(\mathcal{S})$ can be written in the form

$$\Phi = \Phi_0 e^{i\frac{u_L}{\sqrt{N}}\Phi_{\text{eff}}} e^{i\frac{u_R}{\sqrt{N}}\tilde{\Phi}_{\text{eff}}} \tag{5.60}$$

where Φ_0 is neutral under $U, \tilde{U}$ and

$$\begin{aligned} h &= h_0 + \frac{u_L^2}{2N} \\ \tilde{h} &= \tilde{h}_0 + \frac{u_R^2}{2N} \end{aligned} \tag{5.61}$$

However, which operators Φ_0 appear is correlated with $(u_L; u_R)$ in a complicated way. Moreover, $\Phi_{\text{eff}} \to \Phi_{\text{eff}} + 2\pi R/\sqrt{N}$ is not a global symmetry of the theory, but must be accompanied by a discrete transformation on the fields Φ_0.

Equation (5.61) is true for any operators in $\text{Sym}^N(\mathcal{S})$, BPS or not. However, from the BPS mass formula, we can say that (5.61) implies that Φ is BPS iff Φ_0 is BPS.

We can gain some further insight about which charged BPS states do occur by closer examination of (5.30). In the $a = N, d = 1$ term we find a multiplicative factor $Z_\Gamma(N\tau)$. These contribute terms

$$q^{\frac{N}{2}p_L^2}\bar{q}^{\frac{N}{2}p_R^2} = q^{\frac{1}{2N}(Np_L)^2}\bar{q}^{\frac{1}{2N}(Np_R)^2}$$

In the second equality we have written the weights in a form such that, from the BPS formula $h = h_0 + u^2/k$ we can read off the properly normalized charges. This is in accord with the claim that the untwisted sector charge lattice is $N\Gamma(R)$. When $d > 1$ the sum over b induced a nontrivial projection, which in general correlates states with different n_0, m_0 with the charged excitation in a complicated way. The projection condition (5.32) is modified to

$$(n_0 - m_0)(2n_0 + 2m_0 + 1) + nw = 0 \text{ mod } d \tag{5.62}$$

where n, w are related to $(p_L; p_R)$ by (5.57). The spectrum of (bps, bps) states in the $N = ad$ sector is of the form:

$$\begin{aligned} h &= \frac{a^2(4n_0+1)^2}{8N} + N_L + \frac{1}{2N}(ap_L)^2 \\ \tilde{h} &= \frac{a^2(4m_0+1)^2}{8N} + N_R + \frac{1}{2N}(ap_R)^2 \end{aligned} \tag{5.63}$$

where $N_L, N_R \in \mathbb{Z}_+$, and $h - \tilde{h} \in \mathbb{Z}$ is equivalent to (5.62). For $d > 1$ there is a nontrivial correlation, given by (5.62) of which Φ_0 can occur with which Narain vectors.

All charged (bps, bps) states have gaps relative to uncharged states of the form

$$\begin{aligned} h &= h_0 + \frac{p_L^2}{2N} \\ \tilde{h} &= \tilde{h}_0 + \frac{p_R^2}{2N} \end{aligned} \tag{5.64}$$

where $(p_L; p_R)$ is *some* vector in $\Gamma(R)$. This should be carefully distinguished from the statement that for *any* $(p_L; p_R)$ and Φ_0 there is a corresponding Φ with gap (5.64). Indeed, we want to stress that no local quantization of a single Gaussian field can give a spectrum of conformal weights of the form

$$(h, \tilde{h}) = (\frac{p_L^2}{2N}, \frac{p_R^2}{2N}) \tag{5.65}$$

The spectrum (5.65) for $(p_L; p_R) \in \Gamma(R)$ is not equivalent to the spectrum obtained from any Narain lattice at any other radius R'.

Finally, let us note that the contribution of the charged states to the index can be systematically discussed by using the extended definition of Z in (3.23). In terms of the $\mathcal{S}$ theory we replace

$$Z_\Gamma \to Z_\Gamma(\tau, \chi) := \sum_{p \in \Gamma(R)} e^{i\pi\tau p_L^2 - i\pi\bar{\tau} p_R^2 + 2\pi i \chi \cdot p} \tag{5.66}$$

Note that (5.66) is a Siegel-Narain theta function for the lattice $II^{1,1}$ embedded in $\mathbb{R}^{1,1}$ according to (5.57).

Now, in evaluating the indices I_1 and I_2, Z_Γ enters through the combination

$$\frac{1}{d} \sum_{b=0}^{d-1} e^{2\pi i \nu \frac{b}{d}} Z_\Gamma\Big(\frac{a\tau + b}{d}, a\chi\Big) \tag{5.67}$$

for various values of ν. We claim that (5.67) can be expressed in terms of Siegel-Narain theta functions of higher level. We can write (5.67) as

$$\sum_{m,n} \delta_{mn = -\nu \bmod d} \cdot \exp\Big[i\pi\tau \frac{1}{N}(ap_L)^2 - i\pi\bar{\tau}\frac{1}{N}(ap_R)^2 + 2\pi i a \chi \cdot p\Big] \tag{5.68}$$

where $(p_L; p_R) = me + nf$ and $e = (1/R; 1/R)$, $f = \frac{1}{2}(R; -R)$. We can view this as a sum over lattice vectors:

$$v = \frac{a}{\sqrt{N}} p = \frac{a}{\sqrt{N}} me + \frac{a}{\sqrt{N}} nf \tag{5.69}$$

where $mn = -\nu \bmod d$. Let (m_0^i, n_0^i) run over the the finite set of distinct solutions of this congruence, modulo d. Then the general solution is $m = m_0^i + \ell_1 d, n = n_0^i + \ell_2 d$ where $\ell_1, \ell_2 \in \mathbb{Z}$. Thus, we define

$$\Lambda = \sqrt{N} e \mathbb{Z} + \sqrt{N} f \mathbb{Z} = \sqrt{N} \Gamma(R) \tag{5.70}$$

and note that $v = \beta + \lambda$, $\lambda \in \Lambda$, and $\beta \in a\Lambda^*$ satisfies

$$\frac{N}{2} \beta^2 = -a^2 \nu \bmod (aN) \tag{5.71}$$

We can thus write (5.67) in terms of Siegel-Narain theta functions for the lattice Λ. Including the exponential prefactor in (3.23) we have

$$e^{\frac{N\pi}{2\tau_2}(\chi_L^2 + \chi_R^2)} \frac{1}{d} \sum_{b=0}^{d-1} e^{2\pi i \nu \frac{b}{d}} Z_\Gamma\Big(\frac{a\tau + b}{d}\Big) = \sum_{\beta \in a\Lambda^*/\Lambda} \Pi_a(\beta, \nu) \Theta_\Lambda(\tau, 0, \beta; P; \sqrt{N}\chi) \tag{5.72}$$

where $\Pi_a(\beta, \nu)$ projects onto solutions of (5.71) and $\Theta_\Lambda(\tau, \alpha, \beta; P; \xi)$ is a Siegel-Narain theta function for Λ. It follows that we can summarize the χ-dependence in I_2 according to

$$I_2 = \sum_{\beta \in \Lambda^*/\Lambda} I_2^\beta \Theta_\Lambda(\tau, 0, \beta; P; \sqrt{N}\chi) \tag{5.73}$$

where I_2^β are χ-independent and can be written in terms of level k theta functions of $\omega, \tilde{\omega}$, in expressions generalizing I_2^0 in (5.1). (This was our motivation for introducing (3.24) above.) Using (5.72) one finds that similar results hold for I_1. These expressions are of interest in the AdS/CFT context because in the holographic dual theory, the $U(1) \times U(1)$ Chern-Simons sector at level Q_1 is naturally expressed as a linear combination of level Q_1 Siegel-Narain theta functions [18,34].

5.7. Moduli of the $\mathrm{Sym}^N(\mathcal{S})$ *theory*

In [18] criteria for operators to induce moduli spaces of theories with $\mathcal{A}_\gamma$ symmetry are investigated. We need a short multiplet which contains a state with $h = \tilde{h} = 1$ and which is a singlet under the R-symmetry $SU(2)^4$. This only occurs as a state in the short NS representations $(0,0)$ and $(\frac{1}{2}, \frac{1}{2})$ of $\mathcal{A}_\gamma$.

Examination of the index yields two such moduli. First, $\sum_i \partial\varphi(i)\overline{\partial}\varphi(i)$ is the modulus corresponding to deformations of the radius R of the $\mathcal{S}$ theory. This is the state in the $(\frac{1}{2}, \frac{1}{2})$ representation, predicted by the index. Second, the vacuum representation $(0,0)$

is always present. This leads to a universal modulus $U_{-1}\tilde{U}_{-1}|0\rangle$, that is $\sim \partial\Phi_{\rm eff}\overline{\partial}\Phi_{\rm eff}$. Perturbation by this operator formally deforms the radius of $\Phi_{\rm eff}$, but we have seen that at the orbifold point the radius of $\Phi_{\rm eff}$ is not independent of R! In fact, the deformation by this operator does not preserve the structure of a symmetric product orbifold (as is easily confirmed by studying first order conformal perturbation theory).

It is important to recognize that some BPS states exist at the symmetric product point, but do not contribute to the index. Among these there is a potential modulus in the $n = 3$ term in (5.13). Nevertheless, we can invoke the general theorem proved in [18] to see that such a state is a true modulus.[5] This state therefore survives deformations as a BPS state, even though it does not contribute to the index!

Finally, there are potential extra moduli arising from charged BPS states when the radius R takes special values. For example if $p_L^2 = p_R^2$ we get extra contributions to the spectrum. At special radii these can give moduli, e.g. if $u^2 = k$ then the massless NS rep $(h = u^2/k = 1, \ell^+ = 0, \ell^- = 0)_S$ can represent a modulus.

Thus, we conclude that the $\mathrm{Sym}^N(\mathcal{S})$ theory generically has three moduli, only two of which are detected by the index.

5.8. Comments on N not prime

5.8.1. *Computation of I_2*

As explained above, in the zero charge sector of the $\mathrm{Sym}^N(\mathcal{S})$ theory the index I_2 is given by the sum over factorizations of N. Hence, if N is not prime, there are additional contributions to the index. Our main motivation for doing this computation was the AdS/CFT application discussed in [18]. In particular, we were interested in comparing with low lying supergravity states in the case where N is not prime.

To begin, let us consider a contribution associated with the factorization $N = ad$, where both a and d are prime. It follows from (5.32) that there can be two possibilities: 1) $(n_0 - m_0)$ is divisible by d, or 2) $(2n_0 + 2m_0 + 1)$ is divisible by d. Using the fact that

[5] It might be useful to note that the marginal operator can be written simply as follows. Consider the state $|\epsilon_{\mu\nu\lambda\rho}\psi^\mu\psi^\nu\psi^\lambda\psi^\rho|^2|0\rangle_{NS}$ in the 3-long string sector $\mathcal{H}^{(3)}$. This state corresponds to the marginal deformation.

both m_0 and n_0 take values in the range $0, \ldots, (d-1)$, it is easy to find the allowed values of μ and $\tilde{\mu}$ in each case:

$$\begin{array}{lll} 1) & \mu = a(4n_0+1) & \tilde{\mu} = a(4n_0+1) \\ 2a) & \mu = a(4n_0+1) & \tilde{\mu} = a(2d-(4n_0+1)) \\ 2b) & \mu = a(4n_0+1) & \tilde{\mu} = a(6d-(4n_0+1)) \end{array} \tag{5.74}$$

According to (5.30), these values of $(\mu, \tilde{\mu})$ occur with multiplicity a in the massless spectrum of the $\mathrm{Sym}^N(\mathcal{S})$ theory. They determine multiplicities, $n(\ell^+, \ell^-; \tilde{\ell}^+, \tilde{\ell}^-)$, of massless representations in the zero charge sector via (3.14). The resulting contribution to the index is similar to the "long string" contribution (5.33):

$$a \sum_{n=0}^{d-1} |\Theta^-_{a(4n+1),k}|^2 + 2a \sum_{n=0,\mu\neq N}^{d-1} \Theta^-_{a(4n+1),k} \overline{\Theta^-_{2N-a(4n+1),k}} \tag{5.75}$$

where writing the last term we used $\Theta^-_{\mu,k} = \Theta^-_{4N+\mu,k}$. Notice, in particular, that the index I_2 requires states in the NS spectrum with small values of $(\ell^+ - \ell^-)$, which might possibly modify the supergravity spectrum when N is not prime. To see this, let's start with a massless state[6] (ℓ^+, ℓ^-) in the NS sector with small values of ℓ^+ and ℓ^-. Under the spectral flow, this state transforms into a state $(N/2 - \ell^+, \ell^- + 1/2)$. According to (3.14), it corresponds to

$$\mu = \tilde{\mu} = N + 2(\ell^- - \ell^+)$$

Since $\ell^\pm$ are assumed to be small compared to N, we have $\mu = \tilde{\mu} \simeq N$, which is allowed according to (5.74). Therefore, the index I_2 alone does not rule out a possibility that there are extra supergravity states when N is not prime. In order to obtain more information, let us look at the more refined index I_1.

5.8.2. *Computation of* I_1

Generalizing the results for I_1 takes more effort. Let us start with the contribution from the factorization $N = ad$ with $\mu = \tilde{\mu}$, i.e. the contributions with $n_0 = m_0$ in (5.32). In this case, $\mu = a(4n_0 + 1)$, and we are looking at BPS representations with $\tilde{\ell}^+ + \tilde{\ell}^- = (\mu + 1)/2$. We expand Ξ_μ extracted from (5.34) in a q expansion, and find, from the BPS bound $\tilde{L}_0 - c/24 = \mu^2/(4k)$ corresponding to the n^{th} term in the expansion

[6] Here, we consider only the left sector. The right sector is completely similar.

with $n = \frac{1}{2}(2n_0)(2n_0+1)$. In order to obtain the corresponding spectrum of massless representations, we need to find the coefficient of the $(u-v)$ term in the decomposition of $\chi_{R_n}(z_+^a, z_-^a)$ in terms of $SU(2) \times SU(2)$ characters. After some algebra, we find

$$\begin{aligned}\chi_{R_n}(z_+^a, z_-^a) = (u-v) \sum_{\alpha+\beta=a-1} \Big(& \chi_{an_0+\alpha/2}(z_+)\chi_{an_0+\beta/2}(z_-) \\ & - \chi_{an_0+\alpha/2-1/2}(z_+)\chi_{an_0+\beta/2-1/2}(z_-)\Big) \bmod (u-v)^2\end{aligned} \tag{5.76}$$

which leads to the following spectrum of massless representations:

$$\bigoplus_{n_0=0}^{d-1} \bigoplus_{\alpha+\beta=(a-1)} \Big|(an_0 + (\alpha+1)/2, an_0 + (\beta+1)/2)\Big|^2 \tag{5.77}$$

As before, this spectrum is not unique; this is related to the non-uniqueness of the coefficient of $(u-v)$ in eq. (5.76). We chose the particular form of (5.76) because it is manifestly symmetric under exchange $z_+ \leftrightarrow z_-$, and because the resulting spectrum (5.77) is a natural generalization of what we found for $a = N, d = 1$ and $a = 1, d = N$.

The states (5.77) have conformal weights

$$h = \frac{N}{8} + \frac{a^2(4n_0+1)^2}{8N}. \tag{5.78}$$

Upon spectral flow to the NS sector these map to states $|(j_+, j_-)|^2$ with

$$\begin{aligned} j^+ - j^- &= \frac{1}{2}(N-\mu) \\ h &= \frac{1}{4}(N + a - 2\alpha - 2) + \frac{1}{8N}(N-\mu)^2 \qquad 0 \le \alpha \le a-1 \end{aligned} \tag{5.79}$$

where $\mu = a \bmod 4a$, i.e. $\mu = a, 5a, 9a, \ldots, 4N-3a$. Note we can write the conformal weight as

$$h = \frac{(N-1)}{4} + \frac{(N-\mu)^2}{8N} + \frac{a-(\alpha+1)}{2} - \frac{a-1}{4} \tag{5.80}$$

Comparison with (5.9) shows that this is in a sense a composite of the short string and long string spectrum. The short string spectrum is the contribution $\frac{a-(\alpha+1)}{2}$ for $0 \le \alpha \le a-1$. This adds support to the picture that $V(p)$ adds shorts strings to a background of long strings created by $U(p)$.

There are other contributions in the $N = ad$ term, from off-diagonal terms with $\mu \neq \tilde{\mu}$. Consider the (a, d)-contribution to Ξ_μ:

$$\begin{aligned}\Xi_{a(4n_0+1)} &\equiv \sum_{\substack{\ell^+ + \ell^- = (\mu+1)/2 \\ \tilde{\ell}^+, \tilde{\ell}^-}} (-1)^{2\ell^- + 1} n(\ell^\pm; \tilde{\ell}^\pm)\ \mathrm{SCh}_{(\tilde{\ell}^+, \tilde{\ell}^-)} + \text{ massive} = \\ &= \sum_{n \in \mathcal{N}} q^{\frac{a}{d}(n+1/8)} \chi_{R_n}(z_+^a, z_-^a)\end{aligned} \tag{5.81}$$

Here, $\mathcal{N}$ is a set of triangular numbers which satisfy the following relation:

$$n = n_0(2n_0 + 1) \mod d \tag{5.82}$$

Writing $n = m(m+1)/2$, from (5.82) we get

$$\frac{1}{2}(m - 2n_0)(m + 2n_0 + 1) = 0 \mod d \tag{5.83}$$

As in the previous case, our goal here is to describe the set $\mathcal{N}$ explicitly. As in (5.74), when d is prime ($d \neq 2$) we find two groups of solutions corresponding to the two factors in (5.83):

$$\begin{aligned}&1) \qquad m = 2n_0 + sd \qquad s = -1, 0, 1, \ldots, (a-1) \\ &2) \qquad m = sd - 2n_0 - 1 \qquad s = 1, 2, \ldots, (a+1)\end{aligned} \tag{5.84}$$

where, depending on the value of s, n_0 runs over

$$\begin{aligned}&1) \qquad \max\left(0, -\frac{sd}{2}\right) \leq n_0 \leq \min\left(d-1, \frac{(a-s)d-1}{2}\right) \\ &2) \qquad \max\left(0, \frac{(s-a)d}{2}\right) \leq n_0 \leq \min\left(d-1, \frac{sd-1}{2}\right)\end{aligned} \tag{5.85}$$

Notice, that except for the largest or smallest values of s, n_0 takes values in its natural range $0 \leq n_0 \leq (d-1)$.

Therefore, we can label the elements of the set $\mathcal{N}$ by paris (n_0, s) in the range (5.84) – (5.85). Using the decomposition (5.76) of $\chi_{R_n}(z_+^a, z_-^a)$ in terms of the $SU(2) \times SU(2)$ characters:

$$\chi_{R_n}(z_+^a, z_-^a) \to \bigoplus_{\alpha+\beta=(a-1)} \left(\frac{am+\alpha+1}{2}\ ,\ \frac{am+\beta+1}{2}\right) \tag{5.86}$$

and substituting it into (5.81), we find that the simplest spectrum consistent with the index I_1 is, *cf.* (5.77),

$$\bigoplus_{\substack{(n_0,s) \\ \alpha+\beta=(a-1)}} \left(\frac{2an_0+\alpha+1}{2}\ ,\ \frac{2an_0+\beta+1}{2}; \frac{a(2n_0+sd)+\alpha+1}{2}\ ,\ \frac{a(2n_0+sd)+\beta+1}{2}\right) \tag{5.87}$$

$$\bigoplus_{\substack{(n_0,s)\\ \alpha+\beta=(a-1)}} \left(\frac{2an_0+\alpha+1}{2}\ ,\ \frac{2an_0+\beta+1}{2};\frac{a(sd-2n_0-1)+\alpha+1}{2}\ ,\ \frac{a(sd-2n_0-1)+\beta+1}{2}\right)$$

Note, that this spectrum does not contain states with $\ell^+ \simeq N/2$ and $\ell^- \simeq 1$, which under spectral flow might become a part of the supergravity spectrum. The reason is simply that the difference $(\ell^+ - \ell^-)$ for the states (5.87) can not exceed $(\alpha+\beta)/2 \simeq a/2 \leq N/4$.

Finally, we point out that, when $d=2$, the analysis is very similar. Namely, one finds that the spectrum of massless states can still be described by (5.87) where only even values of s appear.

Evaluation for $N = p^r$

Now let us discuss the case when a and d are not relatively prime. In particualar, we will consider the situation when a and d contain several prime factors. The simplest example of this is when N is of the form:

$$N = p^r \tag{5.88}$$

Then, the index can be written as a sum over factorizations of N, which in this case involve factors like

$$d = p^\delta \quad , \quad a = p^{r-\delta} \tag{5.89}$$

for some $\delta = 0, \ldots, r$.

As in the case of a and d prime, the left index I_1 has the form (5.34), where a and d are now given by (5.89) and

$$\mu = p^{r-\delta}(4n_0+1) \tag{5.90}$$

Specifically, we can write (5.34) as a sum over δ

$$I_1^0 = \sum_{\delta=o}^{r} \sum_{n_0=0}^{p^\delta-1} \Theta^-_{\mu,2p^r} \times \frac{1}{p^\delta} \sum_{b=0}^{p^\delta-1} e^{4\pi i \frac{b}{d}(n_0+1/4)^2} \overline{\mathrm{SCh}}^{\mathcal{S},R}\left(\frac{a\tau+b}{d}, z_+^a, z_-^a\right) \tag{5.91}$$

where a contribution from each value of δ looks like

$$\Xi_\mu = \sum_{n\in\mathcal{N}} q^{\frac{a}{d}(n+1/8)} \chi_{R_n}(z_+^a, z_-^a) \tag{5.92}$$

The sum here is over triangular numbers $n = m(m+1)/2$, which satisfy the condition (5.83):

$$\frac{1}{2}(m-2n_0)(m+2n_0+1) = 0 \mod p^\delta \tag{5.93}$$

For the moment let us assume that $p \neq 2$. Then, equation (5.93) implies that the product on the left-hand side is divisible by p^δ. This can happen only if each factor is divisible by a certain power of p:

$$\begin{aligned} &(m-2n_0) \quad \text{divisible by} \quad p^{\delta_1} \\ &(m+2n_0+1) \quad \text{divisible by} \quad p^{\delta_2} \end{aligned} \tag{5.94}$$

such that $\delta_1+\delta_2=\delta$ and $\delta_{1,2}\geq 0$. There are two types of solutions we have to consider:

$$\begin{aligned} &1) \qquad \delta_1=0 \quad \text{or} \quad \delta_2=0 \\ &2) \qquad \delta_1>0 \quad , \quad \delta_2>0 \end{aligned} \tag{5.95}$$

The first case is essentially the same as the one discussed in the previous subsection, when a and d are prime. In particualar, we find the same spectrum (5.87), where the values of a and d are given by (5.89).

The second case in (5.95) describes extra states which appear when a and d are not prime, and this is really what we are after. In this case, we can write (5.94) more explicitly as

$$\begin{aligned} 2m &= -1+s_1p^{\delta_1}+s_2p^{\delta_2} \\ 4n_0 &= -1-s_1p^{\delta_1}+s_2p^{\delta_2} \end{aligned} \tag{5.96}$$

In particular, these equations imply $p|(2m+1)$ and $p|(4n_0+1)$.

Notice, that for $p=2$, instead of (5.94) we find that either $(m-2n_0)$ or $(m+2n_0+1)$ should be divisible by $2^{\delta+1}$. There is no mixed case like 2) in (5.95) because $(m-2n_0)$ and $(m+2n_0+1)$ can not be both even. Therefore, we conclude that, for $p=2$, there are no extra massless states in the spectrum, except for those already describes in (5.87).

Hence, without loss of generality, in what follows we consider $p\neq 2$, which is the case when extra massless states *can* occur. In order to identify these states, we need to find integer solutions to (5.96). First, for given δ_1 and δ_2, let us consider the range of values of s_1 and s_2, which are consistent with the bounds on m and n_0: $0\leq n_0\leq d-1$ and $0\leq m\leq ad-1$. From (5.96) we find

$$\begin{aligned} -2p^{\delta-\delta_1} \leq s_1 \leq p^{r-\delta_1}-1 \\ 1\leq s_2\leq 2p^{\delta-\delta_2}+p^{r-\delta_2}-1 \end{aligned} \tag{5.97}$$

Therefore, we conclude that s_1 and s_2 in this range, which solve (5.96) for integer values of m and n_0, lead to extra massless states of the form

$$\bigoplus_{\substack{(n_0,\delta_{1,2},s_{1,2}) \\ \alpha+\beta=(a-1)}} \left(\frac{2an_0+\alpha+1}{2}\ ,\ \frac{2an_0+\beta+1}{2};\frac{am+\alpha+1}{2}\ ,\ \frac{am+\beta+1}{2}\right) \tag{5.98}$$

Note that, as in the case of a and d prime, these states can not have $\ell^+ \simeq N/2$ and $\ell^- \simeq 1$, which would make them visible in the supergravity spectrum. In order to describe the set of values (s_1, s_2) more explicitly, we need to analyze the congruence of (5.96) mod 2 and mod 4.

6. Computation of the index for the $U(2)_r$ theories

We will now consider a generalization to the theories

$$Sym^N(\mathcal{S}_r) \tag{6.1}$$

where $\mathcal{S}_r$ is a product of a bosonic WZW $U(2)_r$ with 4 MW fermions. As discovered in [26,12] this theory has $\mathcal{A}_\gamma$ symmetry with $(k^+, k^-) = (r+1, 1)$ and $k = r+2$. In this section we present some partial results on the index for this theory.

The R-sector supercharacter before GSO projection is

$$Z = \sum_{j=0}^{r/2} \left| \chi_j^{(r)}(\tau, \omega_+) \mathrm{SCh}^{\mathcal{S},R}(\tau, \omega_+, \omega_-) \right|^2 \sum_{\Gamma^{1,1}} q^{\frac{1}{2}p_L^2} \overline{q}^{\frac{1}{2}p_R^2} \tag{6.2}$$

It is worth computing the index first for the theory $\mathcal{S}_r$. Using (5.20), identity (D.7) and the formula for Kac-Moody characters:

$$\chi_j^{(r)}(\omega, \tau) = \frac{\Theta^-_{2j+1, r+2}(\omega, \tau)}{\Theta^-_{1,2}(\omega, \tau)} \tag{6.3}$$

we immediately find for the left-index:

$$I_1(\mathcal{S}_r) = \sum_{j=0}^{r/2} \Theta^-_{2j+1, r+2}(\omega, \tau) \overline{\Xi_j} \sum_{\Gamma^{1,1}} q^{\frac{1}{2}p_L^2} \overline{q}^{\frac{1}{2}p_R^2} \tag{6.4}$$

where in the charge zero sector

$$\Xi_\mu = \chi_j^{(r)}(\tau, \tilde{\omega}_+) \mathrm{SCh}^{\mathcal{S},R}(\tau, \tilde{\omega}_+, \tilde{\omega}_-) \tag{6.5}$$

for $\mu = 2j+1$.

Consider the term corresponding to j in (6.4). Now, as usual, we have to worry that the index in principle receives contributions from any representation (ℓ^+, ℓ^-) with $2(\ell^+ + \ell^-) - 1 = 2j+1$ on the left. Therefore, we need to examine the remaining character

on the right. Again applying the index to Ξ_j we get $\Theta^-_{2j+1,r+2}$, so the only massless representations which can occur in the expansion of (6.5) have $2(\ell^+ + \ell^-) - 1 = 2j+1$. Now we compute q expansions to get

$$\Xi_\mu = q^{(j+1/2)^2/(r+2)}(\tilde{u} - \tilde{v})\chi_j(\tilde{z}_+) + \cdots \tag{6.6}$$

showing that only the representation $(j+1/2, 1/2)$ actually appears.

It follows that the only massless representation on the right which appears is $(j + 1/2, 1/2)$. Therefore, the simplest consistent solution is

$$Z = \sum_{j=0}^{r/2} \Big| \mathrm{SCh}(j+1/2, 1/2) \Big|^2 + \cdots \tag{6.7}$$

where the dots stand for massive representations.

Thus the RR BPS spectrum of the theory $\mathcal{S}_r$ is

$$(j+1/2, 1/2; j+1/2, 1/2) \qquad 0 \le j \le r/2 \tag{6.8}$$

Under spectral flow to the NSNS sector we get $(j,0;j,0)$ for $0 \le j \le r/2$.

Let us now turn to the symmetric product (6.1). A small computation shows that the left-index is

$$I_1(Sym^N(\mathcal{S}_r)) = -z_+ \frac{d}{dz_-}(T_N Z) = \sum_{a|N} \sum_{n_0=0}^{d-1} \sum_{j=0}^{r/2} \Theta_{\mu,K}(\omega,\tau)\overline{\Xi_{j,n_0}} \tag{6.9}$$

where

$$\begin{aligned} \mu &= a(2n_0(r+2) + 2j + 1) \\ K &= N(r+2) \end{aligned} \tag{6.10}$$

$$\Xi_{j,n_0} := \frac{1}{d} \sum_{b=0}^{d-1} e^{-2\pi i \frac{b}{d}(r+2)\left(n_0 + \frac{2j+1}{2r+4}\right)^2} \left(\chi_j^{(r)} \mathrm{SCh}^{\mathcal{S},R}\right)\left(\frac{a\tau+b}{d}, a\tilde{\omega}_+, a\tilde{\omega}_-\right) Z_\Gamma\left(\frac{a\tau+b}{d}\right) \tag{6.11}$$

Taking another derivative to compute the index I_2 we find, in the charge zero sector the projection operator constraining constraint $\tilde{\mu} = a((2r+4)m_0 + 2j + 1) = 2(\tilde{\ell}^+ + \tilde{\ell}^-) - 1$ is

$$(m_0 - n_0)\Big((r+2)(m_0+n_0) + 2j + 1\Big) = 0 \bmod d \tag{6.12}$$

This generalizes (5.32).

Again, we can analyse the terms $a = 1, d = N$ and $a = N, d = 1$ as before.

6.1. Analysis of $a = N, d = 1$

Now we attempt to decompose

$$\chi_j^{(r)} \mathrm{SCh}^{\mathcal{S},R}(N\tau, N\omega_+, N\omega_-) \tag{6.13}$$

in terms of massless characters. We note that the q-expansion of the $\mathcal{S}_r$ theory begins as

$$\chi_j^{(r)} \mathrm{SCh}^{\mathcal{S},R}(\tau, \omega_+, \omega_-) = e^{2\pi i \tau \frac{(j+1/2)^2}{r+2}} \chi_j(z_+)(\chi_{1/2}(z_+) - \chi_{1/2}(z_-)) + \cdots \tag{6.14}$$

Now, let us work under the hypothesis that the only BPS states which contribute come from this leading term in the q expansion. It then follows from the BPS bound that the only spins which appear satisfy:

$$\ell^+ + \ell^- = Nj + \frac{N+1}{2} \tag{6.15}$$

It is not difficult to prove the generalization of (5.38):

$$\begin{aligned} \chi_j(z_+^N)\big(\chi_{1/2}(z_+^N) - \chi_{1/2}(z_-^N)\big) = (u-v) \sum_{a+b=N-1} \Big[&\chi_{Nj+a/2}(z_+)\chi_{b/2}(z_-) \\ - \chi_{Nj+(a-1)/2}(z_+)\chi_{(b-1)/2}(z_-) \Big] &+ (u-v)^2 \chi_R(z_+, z_-) \end{aligned} \tag{6.16}$$

where the second term has a second order zero at $u = v$ and corresponds to the leading q-expansion of a massive character. Therefore, we conclude that

$$\chi_j^{(r)} \mathrm{SCh}^{\mathcal{S},R}(N\tau, N\omega_+, N\omega_-) = \sum_{a+b=N-1} (-1)^{N-a} \mathrm{SCh}(Nj + (a+1)/2, (N-a)/2) + \cdots \tag{6.17}$$

where we are using supercharacters of $\mathcal{A}_\gamma$ at level $(k^+ = N(r+1), k^- = N)$. The simplest spectrum consistent with the index is

$$\bigoplus_{j=0}^{r/2} \bigoplus_{a=0}^{N-1} \Big| (Nj + (a+1)/2, (N-a)/2) \Big|^2 . \tag{6.18}$$

Upon spectral flow to the NS sector this is

$$\bigoplus_{j_1=0}^{r/2} \bigoplus_{j_2=0}^{\frac{1}{2}(N-1)} \Big| \big(Nj_1 + j_2, j_2\big)_{NS} \Big|^2 . \tag{6.19}$$

Once again, we remind the reader that this spectrum is not unique; one can add virtual representations of the net index zero.

6.2. Analysis of $a=1, d=N$

Now (6.11) simplifies to

$$\Xi^{ls}_{j,n_0} = \frac{1}{N}\sum_{b=0}^{N-1} e^{-2\pi i \frac{b}{N}(r+2)\left(n_0+\frac{2j+1}{2r+4}\right)^2}\left(\chi_j^{(r)}\mathrm{SCh}^{\mathcal{S},R}\right)\left(\frac{\tau+b}{N},\tilde{\omega}_+,\tilde{\omega}_-\right) \tag{6.20}$$

In order to expand this expression in terms of supercharacters, first we would need to write it as a power series in q. It is convenient to introduce

$$S_j := \chi_j^{(r)}\mathrm{SCh}^{\mathcal{S},R} = q^{\frac{(2j+1)^2}{4(r+2)}}\sum_{n=0}^{\infty} q^n s_{j,n}(z_+,z_-) \tag{6.21}$$

where $s_{j,n}(z_+,z_-)$ are certain combinations of $SU(2)\times SU(2)$ characters. Then, as in the previous section, one can easily perform the sum over b in (6.20). This leads to

$$\Xi^{ls}_{j,n_0} = \sum_{\substack{n=0 \\ n=n_0((r+2)n_0+2j+1) \mod N}}^{\infty} q^{\frac{1}{N}\left(n+\frac{(2j+1)^2}{4(r+2)}\right)} s_{j,n}(z_+,z_-) \tag{6.22}$$

which is a generalization of (5.46). Unfortunately, the explicit calculation of $s_{j,n}(z_+,z_-)$ is very difficult for generic values of r and $j=0,\ldots,r/2$. However, we analyzed (6.22) in a number of examples. Curiously, in all the cases that we studied, Ξ^{ls}_{j,n_0} can be expanded in terms of supercharacters with non-negative multiplicities.

Example. $N=5,\ r=2,\ j=1,\ n_0=2$

$$\begin{aligned}
\Xi^{ls}_{1,2} = {} & \mathrm{SCh}(9/2,3/2)+ \\
& + 2\mathrm{SCh}_{\mathrm{m}}(41/80,5/2,2/2) + \mathrm{SCh}_{\mathrm{m}}(41/80,4/2,3/2)+ \\
& + 3\mathrm{SCh}_{\mathrm{m}}(41/80,3/2,2/2) + 3\mathrm{SCh}_{\mathrm{m}}(121/80,9/2,2/2)+ \\
& + 11\mathrm{SCh}_{\mathrm{m}}(121/80,8/2,3/2) + 7\mathrm{SCh}_{\mathrm{m}}(121/80,7/2,4/2)+ \\
& + \mathrm{SCh}_{\mathrm{m}}(121/80,6/2,5/2) + 36\mathrm{SCh}_{\mathrm{m}}(121/80,7/2,2/2)+ \\
& + 63\mathrm{SCh}_{\mathrm{m}}(121/80,6/2,3/2) + 32\mathrm{SCh}_{\mathrm{m}}(121/80,5/2,4/2)+ \\
& + 3\mathrm{SCh}_{\mathrm{m}}(121/80,4/2,5/2) + 115\mathrm{SCh}_{\mathrm{m}}(121/80,5/2,2/2)+ \\
& + 123\mathrm{SCh}_{\mathrm{m}}(121/80,4/2,3/2) + 36\mathrm{SCh}_{\mathrm{m}}(121/80,3/2,4/2)+ \\
& + 2\mathrm{SCh}_{\mathrm{m}}(121/80,2/2,5/2) + 124\mathrm{SCh}_{\mathrm{m}}(121/80,3/2,2/2)+ \\
& + 66\mathrm{SCh}_{\mathrm{m}}(121/80,2/2,3/2) + \ldots
\end{aligned} \tag{6.23}$$

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Appendix A. Leading q expansion of the Peterson-Taormina characters

A.1. Massive R-sector characters

The unitarity range in the R sector is[7]

$$1 \leq \ell^{\pm} < \frac{1}{2}k^{\pm} \tag{A.1}$$

and massive representations satisfy:

$$h - c/24 \geq u^2/k + (\ell^+ + \ell^- - 1)^2/k \tag{A.2}$$

In the unitarity range, the q expansion is always:

$$\mathrm{SCh}_{\mathrm{massive}}(h, \ell^{\pm}) = (-1)^{2\ell^-} q^{h-c/24} (\chi_{\frac{1}{2}}(z_+) - \chi_{\frac{1}{2}}(z_-))^2 \chi_{\ell^+-1}(z_+)\chi_{\ell^--1}(z_-) + \dots \tag{A.3}$$

A.2. Massless R-sector characters

The unitary range is now:

$$\frac{1}{2} \leq \ell^{\pm} \leq \frac{1}{2}k^{\pm} \tag{A.4}$$

$$h - c/24 = u^2/k + (\ell^+ + \ell^- - 1/2)^2/k \tag{A.5}$$

For the massless characters we find the leading term is given by

$$\begin{aligned} &\mathrm{SCh}(\ell^+, \ell^-) = (-1)^{2\ell^- -1} q^{h-c/24} \cdot \\ &\cdot (\chi_{\frac{1}{2}}(z_+) - \chi_{\frac{1}{2}}(z_-)) \Big(\chi_{\ell^+-1/2}(z_+)\chi_{\ell^--1/2}(z_-) - \chi_{\ell^+-1}(z_+)\chi_{\ell^--1}(z_-) \Big) + \dots \end{aligned} \tag{A.6}$$

Here it is understood that $\chi_{-1/2} = 0$.

[7] [14] eq. 2.9 and [15] eq. 2.18 disagree on the upper limit. We take [15] eq. 2.18.

A.3. Massive NS-sector characters

Unitarity bounds are:[8]

$$0 < \ell^{\pm} < \frac{1}{2}(k^{\pm} - 1) \tag{A.7}$$

with

$$kh > (\ell^+ - \ell^-)^2 + k^-\ell^+ + k^+\ell^- + u^2 \tag{A.8}$$

Note this region is empty for $k^{\pm} = 1$. In that case there are only massless characters.

The leading term in the q expansion is:

$$\mathrm{SCh}^{\mathcal{A}_\gamma, NS}_{\mathrm{massive}} = (-1)^{2\ell^-} q^{h-c/24} \chi_{\ell^+}(z_+)\chi_{\ell^-}(z_-) + \dots \tag{A.9}$$

A.4. Massless NS-sector characters

Unitarity bounds are:

$$0 \leq \ell^{\pm} \leq \frac{1}{2}(k^{\pm} - 1) \tag{A.10}$$

with

$$kh = (\ell^+ - \ell^-)^2 + k^-\ell^+ + k^+\ell^- + u^2 \tag{A.11}$$

The leading term in the q expansion is the *same* for the massless and massive case and is:

$$\mathrm{SCh}^{\mathcal{A}_\gamma, NS}(\ell^+, \ell^-) = (-1)^{2\ell^-} q^{h-c/24} \chi_{\ell^+}(z_+)\chi_{\ell^-}(z_-) + \cdots \tag{A.12}$$

The massless and massive characters differ at $q^{h+1/2}$.

Appendix B. Characters of the Large $\mathcal{N} = 4$ SCA

The character of a *massive* representation in the Ramond sector of the $\mathcal{A}_\gamma$ is given by [14]:

$$\begin{aligned}
\mathrm{Ch}^{\mathcal{A}_\gamma, R}(k^{\pm}, h, \ell^{\pm}; q, z_{\pm}) &= q^{h-c/24} \left(F^R(q, z_+, z_-)\right)^2 (1 + z_+^{-1} z_-^{-1})^2 (1 + z_+^{-1} z_-)^2 \\
&\times \prod_{n=1}^{\infty} (1-q^n)^{-4} (1 - z_+^2 q^n)^{-1} (1 - z_+^{-2} q^{n-1})^{-1} (1 - z_-^2 q^n)^{-1} (1 - z_-^{-2} q^{n-1})^{-1} \\
&\times \sum_{m=-\infty}^{+\infty} q^{m^2 k^+ + (2\ell^+ - 1)m} \left(z_+^{2(mk^+ + \ell^+)} - z_+^{-2(mk^+ + \ell^+ - 1)} \right) \\
&\times \sum_{n=-\infty}^{+\infty} q^{n^2 k^- + (2\ell^- - 1)n} \left(z_-^{2(nk^- + \ell^-)} - z_-^{-2(nk^- + \ell^- - 1)} \right)
\end{aligned} \tag{B.1}$$

[8] Again, the lower limit differs from [14], eq. 2.10. Their statement is not consistent with spectral flow.

where

$$F^R(q,z_\pm) = \prod_{n\in\mathbb{Z}_+}(1+z_+z_-q^n)(1+z_+^{-1}z_-^{-1}q^n)(1+z_+z_-^{-1}q^n)(1+z_+^{-1}z_-q^n) \tag{B.2}$$

The authors of [14][15] mostly work with $\tilde{\mathcal{A}}_\gamma$ which is obtained from $\mathcal{A}_\gamma$ by removing a free boson and four fermions. The quantum numbers of representations of these two algebras are related via, see *e.g.* [13],

$$\begin{array}{ccc} \underline{\mathcal{A}_\gamma} & & \underline{\tilde{\mathcal{A}}_\gamma} \\ c & \to & \tilde{c} = c-3 \\ k_\pm & \to & \tilde{k}_\pm = k_\pm - 1 \\ \ell^\pm_{NS} & \to & \tilde{\ell}^\pm_{NS} = \ell^\pm_{NS} \\ \ell^\pm_R & \to & \tilde{\ell}^\pm_R = \ell^\pm_R - \frac{1}{2} \end{array} \tag{B.3}$$

and, similarly, for the characters:

$$\mathrm{Ch}^{\mathcal{A}_\gamma,R}(k^\pm,h,\ell^\pm;q,z_\pm) = \mathrm{Ch}^{\mathcal{S},R}(q,z_\pm)\times \mathrm{Ch}^{\tilde{\mathcal{A}}_\gamma,R}(\tilde{k}^\pm,\tilde{h},\tilde{\ell}^\pm;q,z_\pm) \tag{B.4}$$

where

$$\mathrm{Ch}^{\mathcal{S},R}(q,z_\pm) = q^{u^2/k+1/8}F^R(q,z_\pm)\times\prod_{n=1}^{\infty}(1-q^n)^{-1}(1+z_+^{-1}z_-^{-1})(1+z_+^{-1}z_-)z_+ \tag{B.5}$$

Notice, that (B.5) vanishes when $z_+ = -z_-$. From (B.4) it follows that $\mathrm{Ch}^{\mathcal{A}_\gamma,R}$ also vanishes, as long as $\mathrm{Ch}^{\tilde{\mathcal{A}}_\gamma,R}$ is finite at $z_+ = -z_-$.

In [15], Petersen and Taormina give an independent derivation of the $\tilde{\mathcal{A}}_\gamma$ characters for massive states,

$$\begin{aligned} \mathrm{Ch}^{\tilde{\mathcal{A}}_\gamma,R}(\tilde{k}^\pm,h,\ell^\pm;q,z_\pm) &= q^{h-\tilde{c}/24}F^R(q,z_+,z_-) \\ &\times\prod_{n=1}^{\infty}(1-q^n)^{-2}(1-z_+^2q^n)^{-1}(1-z_+^{-2}q^n)^{-1}(1-z_-^2q^n)^{-1}(1-z_-^{-2}q^n)^{-1} \\ &\times\prod_{n=1}^{\infty}(1-q^n)^{-1}(z_+^{-1}+z_-^{-1})(1+z_+^{-1}z_-^{-1})(1-z_+^{-2})^{-1}(1-z_-^{-2})^{-1} \\ &\times\sum_{m,n=-\infty}^{\infty}q^{n^2k^++2\ell^+n+m^2k^-+2\ell^-m}\sum_{\epsilon_+,\epsilon_-=\pm1}\epsilon_+\epsilon_-z_+^{2\epsilon_+(\ell^++nk^+)}z_-^{2\epsilon_-(\ell^-+mk^-)} \end{aligned} \tag{B.6}$$

which they claim agrees with (B.4). For example, it is easy to check that both (B.6) and (B.5) have a simple zero at $z_+ = -z_-$, whereas (B.1) has a double zero.

For *massless* states the characters look like [15]:

$$\begin{aligned}
\mathrm{Ch}_0^{\tilde{\mathcal{A}}_\gamma,R}(\tilde{k}^\pm, h, \ell^\pm; q, z_\pm) &= q^{h-\tilde{c}/24} F^R(q, z_+, z_-) \\
&\times \prod_{n=1}^{\infty} (1-q^n)^{-2}(1-z_+^2 q^n)^{-1}(1-z_+^{-2}q^n)^{-1}(1-z_-^2 q^n)^{-1}(1-z_-^{-2}q^n)^{-1} \\
&\times \prod_{n=1}^{\infty} (1-q^n)^{-1}(z_+^{-1}+z_-^{-1})(1+z_+^{-1}z_-^{-1})(1-z_+^{-2})^{-1}(1-z_-^{-2})^{-1} \\
&\times \sum_{m,n=-\infty}^{\infty} q^{n^2k^+ + 2\ell^+ n + m^2 k^- + 2\ell^- m} \\
&\times \sum_{\epsilon_+,\epsilon_- = \pm 1} \epsilon_+ \epsilon_- z_+^{2\epsilon_+(\ell^+ + nk^+)} z_-^{2\epsilon_-(\ell^- + mk^-)} (z_+^{-\epsilon_+} q^{-n} + z_-^{-\epsilon_-} q^{-m})^{-1}
\end{aligned} \tag{B.7}$$

Using (B.4) we can obtain the character of the corresponding massless state in the Ramond sector in the $\mathcal{A}_\gamma$ theory. Since (B.7) is finite at $z_+ = -z_-$, the $\mathcal{A}_\gamma$ character of a massless state has only a simple zero at this point.

For completeness, we include here the characters $\mathrm{Ch}^{\mathcal{N}=4}(k, \ell; q, z)$ of the small $\mathcal{N} = 4$ superconformal algebra:

$$\begin{aligned}
\mathrm{Ch}^{\mathcal{N}=4,R}(k,\ell) = z q^{k/4} \prod_{n=1}^{\infty} & \frac{(1+zq^n)^2(1+z^{-1}q^{n-1})^2}{(1-q^n)^2(1-q^n z^2)(1-q^{n-1}z^{-2})} \\
& \times \sum_{m\in\mathbb{Z}} q^{(k+1)m^2+2\ell m} \Big[\frac{z^{2m(k+1)+2\ell-1}}{(1+z^{-1}q^{-m})^2} - \frac{z^{-2m(k+1)-2\ell-1}}{(1+zq^{-m})^2} \Big]
\end{aligned} \tag{B.8}$$

Appendix C. Spectral flow rules

The spectral flow isomorphism involves, among other things:

$$\begin{aligned}
\hat{L}_0 &= L_0 - \rho A_0^{+,3} + \frac{1}{4} k^+ \rho^2 \\
\hat{A}_0^{+,3} &= A_0^{+,3} - \frac{1}{2}\rho k^+ \\
\hat{A}_0^{-,3} &= A_0^{-,3}
\end{aligned} \tag{C.1}$$

We will call this s_+^ρ. Notice ρ has to be an integer since $A_0^{+,3}$ should have half-integer spectrum. If ρ is odd it exchanges NS and R sectors.

There is the corresponding flow on the other $SU(2)$ called s_-^η. Among other things:

$$\begin{aligned}
\hat{L}_0 &= L_0 - \eta A_0^{-,3} + \frac{1}{4} k^- \eta^2 \\
\hat{A}_0^{+,3} &= A_0^{+,3} \\
\hat{A}_0^{-,3} &= A_0^{-,3} - \frac{1}{2}\eta k^-
\end{aligned} \tag{C.2}$$

Spectral flow by $\rho = 1$ or $\eta = 1$ defines operators $s_{\pm}$ exchanging R and NS sector representations. The behavior of the highest weight states under spectral flow is complicated, and hence it is important to distinguish the action of $s_{\pm}$ on states and on representation labels. On the latter we have:

On massive representations:

$$s_+ : (h, \ell^+, \ell^-)_{NS} \leftrightarrow (h - \ell^+ + \frac{1}{4}k^+, \frac{1}{2}k^+ - \ell^+, \ell^- + 1)_R \tag{C.3}$$

On massless representations:

$$\begin{aligned} s_+ :& (h, \ell^+, \ell^-)_{NS} \to (h - \ell^+ + \frac{1}{4}k^+, \frac{1}{2}k^+ - \ell^+, \ell^- + \frac{1}{2})_R \\ & (h, \ell^+, \ell^-)_R \to (h - \ell^+ + \frac{k^+}{4}, \frac{1}{2}k^+ - \ell^+, \ell^- - \frac{1}{2})_{NS} \end{aligned} \tag{C.4}$$

$$\begin{aligned} s_- :& (h, \ell^+, \ell^-)_{NS} \to (h - \ell^- + \frac{1}{4}k^-, \ell^+ + \frac{1}{2}, \frac{1}{2}k^- - \ell^-)_R \\ & (h, \ell^+, \ell^-)_R \to (h - \ell^- + \frac{k^-}{4}, \ell^+ - \frac{1}{2}, \frac{1}{2}k^- - \ell^-)_{NS} \end{aligned} \tag{C.5}$$

Appendix D. Holomorphic level k theta functions

Our convention for the level 1/2 theta functions is:

$$\begin{aligned} \vartheta[{\theta \atop \phi}](0|\tau) &:= \sum q^{\frac{1}{2}(n+\theta)^2} e^{2\pi i (n+\theta)\phi} \\ &= e^{2\pi i \theta\phi} q^{\frac{\theta^2}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 + e^{2\pi i \phi} q^{n - \frac{1}{2} + \theta})(1 + e^{-2\pi i \phi} q^{n - \frac{1}{2} - \theta}) \end{aligned} \tag{D.1}$$

The standard definition of the special theta functions is:

$$\vartheta_1(\omega|\tau) := \vartheta[{1/2 \atop 1/2}](\omega|\tau) \tag{D.2}$$

It has a product representation:

$$\vartheta_1(\omega|\tau) = -2\sin(\pi\omega) q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 - e^{2\pi i \omega} q^n)(1 - e^{-2\pi i \omega} q^n) \tag{D.3}$$

Level k theta functions $\Theta_{\mu,k}(\omega,\tau)$, $\mu=-k+1,\dots,k$ are defined by:

$$\begin{aligned}\Theta_{\mu,k}(\omega,\tau) &= \sum_{\ell\in\mathbb{Z},\ell=\mu \bmod 2k} q^{\ell^2/(4k)} y^\ell \\ &\equiv \sum_{n\in\mathbb{Z}} q^{k(n+\mu/(2k))^2} y^{(\mu+2kn)} \\ &= q^{\frac{\mu^2}{4k}} y^\mu \sum_{n\in\mathbb{Z}} q^{kn^2+n\mu} y^{2kn}\end{aligned} \tag{D.4}$$

where $y=e^{2\pi i\omega}$.

There are $2k$ independent functions which may be split into $(k+1)$ even and $(k-1)$ odd functions of ω.

Note:

$$\begin{aligned}\Theta_{\mu,k}(\omega,\tau) &= \Theta_{\mu+2ka,k}(\omega,\tau) \qquad a\in\mathbb{Z} \\ \Theta_{\mu,k}(-\omega,\tau) &= \Theta_{2k-\mu,k}(\omega,\tau) = \Theta_{-\mu,k}(\omega,\tau)\end{aligned} \tag{D.5}$$

It is useful to define:

$$\Theta^{\pm}_{\mu,k}(\omega,\tau) := \Theta_\mu(\omega,\tau) \pm \Theta_\mu(-\omega,\tau) \tag{D.6}$$

Then:

$$\Theta_{1,2}(\omega,\tau) - \Theta_{-1,2}(\omega,\tau) = -i\vartheta_1(2\omega|\tau) \tag{D.7}$$

To compute Hecke transforms we need

$$\Theta_{\mu,k}\big(a\omega,\frac{a\tau+b}{d}\big) = \sum_{n_0=0}^{d-1} e^{2\pi i\frac{b}{d}k(n_0+\mu/(2k))^2}\Theta_{a(2kn_0+\mu),Mk}(\omega,\tau) \tag{D.8}$$

where $M=ad$. (There is no need to assume a,d are relatively prime.) Notice that one can symmetrize or anti-symmetrize (D.8) under $\omega\to-\omega$.

Modular transformations:

$$\Theta_{\mu,k}(\omega,\tau+1) = e^{2\pi i\frac{\mu^2}{4k}}\Theta_{\mu,k}(\omega,\tau) \tag{D.9}$$

$$\Theta_{\mu,k}(-\omega/\tau,-1/\tau) = (-i\tau)^{1/2} e^{2\pi i k\omega^2/\tau} \sum_{\nu=0}^{2k-1} \frac{1}{\sqrt{2k}} e^{2\pi i\frac{\mu\nu}{2k}}\Theta_{\nu,k}(\omega,\tau) \tag{D.10}$$

It is useful to note the projection onto odd theta functions:

$$\Theta^-_{\mu,k}(-\omega/\tau,-1/\tau) = i(-i\tau)^{1/2} e^{2\pi i k\omega^2/\tau} \sum_{\nu=1}^{k-1} \sqrt{\frac{2}{k}}\sin(\pi\frac{\mu\nu}{k})\Theta^-_{\nu,k}(\omega,\tau) \tag{D.11}$$

Appendix E. Siegel-Narain Theta functions

Let Λ be a lattice of signature (b_+, b_-). Let P be a decomposition of $\Lambda \otimes \mathbb{R}$ as a sum of orthogonal subspaces of definite signature:

$$P : \Lambda \otimes \mathbb{R} \cong \mathbb{R}^{b_+,0} \perp \mathbb{R}^{0,b_-} \tag{E.1}$$

Let $P_\pm(\lambda) = \lambda_\pm$ denote the projections onto the two factors. We also write $\lambda = \lambda_+ + \lambda_-$. With our conventions $P_-(\lambda)^2 \leq 0$.

Let $\Lambda + \gamma$ denote a translate of the lattice Λ. We define the Siegel-Narain theta function

$$\begin{aligned} \Theta_{\Lambda+\gamma}(\tau, \alpha, \beta; P, \xi) \equiv& \exp[\frac{\pi}{2y}(\xi_+^2 - \xi_-^2)] \\ \sum_{\lambda \in \Lambda+\gamma} \exp\Big\{ i\pi\tau(\lambda+\beta)_+^2 + i\pi\overline{\tau}(\lambda+\beta)_-^2 &+ 2\pi i(\lambda+\beta, \xi) - 2\pi i(\lambda + \frac{1}{2}\beta, \alpha)\Big\} \\ =& e^{i\pi(\beta,\alpha)}\exp[\frac{\pi}{2y}(\xi_+^2 - \xi_-^2)] \\ \sum_{\lambda \in \Lambda+\gamma} \exp\Big\{ i\pi\tau(\lambda+\beta)_+^2 + i\pi\overline{\tau}(\lambda+\beta)_-^2 &+ 2\pi i(\lambda+\beta, \xi) - 2\pi i(\lambda + \beta, \alpha)\Big\} \end{aligned} \tag{E.2}$$

where $y = \tau_2$.

The main transformation law is:

$$\Theta_\Lambda(-1/\tau, \alpha, \beta; P, \frac{\xi_+}{\tau} + \frac{\xi_-}{\overline{\tau}}) = \sqrt{\frac{|\Lambda|}{|\Lambda^*|}}(-i\tau)^{b_+/2}(i\overline{\tau})^{b_-/2}\Theta_{\Lambda'}(\tau, \beta, -\alpha; P, \xi) \tag{E.3}$$

where Λ' is the dual lattice. If there is a characteristic vector, call it w_2, such that

$$(\lambda, \lambda) = (\lambda, w_2) \bmod 2 \tag{E.4}$$

for all λ then we have in addition:

$$\Theta_\Lambda(\tau+1, \alpha, \beta; P, \xi) = e^{-i\pi(\beta, w_2)/2}\Theta_\Lambda(\tau, \alpha - \beta - \frac{1}{2}w_2, \beta; P, \xi) \tag{E.5}$$

Appendix F. Symmetric product partition functions

We review the derivation of [19]. The Hilbert space is

$$\mathcal{H}(\mathrm{Sym}^N(\mathcal{C}_0)) = \bigoplus_{(n)^{\ell_n}} \bigotimes_n \mathrm{Sym}^{\ell_n}(\mathcal{H}^{(n)}(\mathcal{C}_0)) \tag{F.1}$$

where we sum over partitions $\sum n\ell_n = N$.

So the generating function is

$$\mathcal{Z} := 1 + \sum_{N\geq 1} p^N \mathrm{Tr}_{\mathcal{H}(\mathrm{Sym}^N(\mathcal{C}_0))} q^H y^J \overline{q}^{\overline{H}} \overline{y}^{\overline{J}} = \prod_{n=1}^{\infty} \sum_{\ell_n=0}^{\infty} p^{n\ell_n} \mathrm{Tr}_{\mathrm{Sym}^{\ell_n}(\mathcal{H}^{(n)})} q^H y^J \overline{q}^{\overline{H}} \overline{y}^{\overline{J}} \quad \text{(F.2)}$$

Here y^J is short for insertions of z_+, z_- etc.

Now the standard formula for traces in symmetric products of vector spaces gives

$$\sum_{\ell_n=0}^{\infty} p^{n\ell_n} \mathrm{Tr}_{\mathrm{Sym}^{\ell_n}(\mathcal{H}^{(n)})} q^H y^J \overline{q}^{\overline{H}} \overline{y}^{\overline{J}} = \prod_{\text{basis } \mathcal{H}^{(n)}} \frac{1}{1 - p^n q^H y^J \overline{q}^{\overline{H}} \overline{y}^{\overline{J}}} \tag{F.3}$$

where we take a product over an eigenbasis in $\mathcal{H}^{(n)}(\mathcal{C}_0)$, the Hilbert space of a string of length n. But

$$\mathrm{Tr}_{\mathcal{H}^{(n)}(\mathcal{C}_0)}(q^H y^J \overline{q}^{\overline{H}} \overline{y}^{\overline{J}}) = \frac{1}{n} \sum_{b=0}^{n-1} \mathrm{Tr}_{\mathcal{H}(\mathcal{C}_0)} \omega^b q^{\frac{1}{n}H} y^J \overline{q}^{\frac{1}{n}\overline{H}} \overline{y}^{\overline{J}} \tag{F.4}$$

where $\omega = e^{2\pi i(L_0 - \overline{L}_0)/n}$, and $H = L_0 - c/24$. Thus the sum on b projects to states that satisfy $\Delta - \overline{\Delta} = 0 \bmod n$. From this the symmetric product formula follows.

If we have a $\mathbb{Z}_2$-graded Hilbert space then we should take a supertrace, and use the rule:

$$\sum_{\ell=0}^{\infty} p^{\ell} \mathrm{STr}_{\mathrm{Sym}^{\ell}(\mathcal{H})}(\mathcal{O}) = \prod_{\text{eigenbasis } \mathcal{H}_0} \frac{1}{(1 - p\mathcal{O}_i)} \prod_{\text{eigenbasis } \mathcal{H}_1} (1 - p\mathcal{O}_i) = \exp\Big[\sum_s \frac{p^s}{s} \mathrm{STr}_{\mathcal{H}}(\mathcal{O}^s)\Big] \tag{F.5}$$

On the other hand, for the Hecke operator formula we take the logarithm of (F.2) using (F.3):

$$\log \mathcal{Z} = \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} \sum_{\text{basis } \mathcal{H}^{(n)}} \frac{1}{s} p^{ns} (q^H y^J \overline{q}^{\overline{H}} \overline{y}^{\overline{J}})^s \tag{F.6}$$

Using again (F.4) this can be written as

$$\log \mathcal{Z} = \sum_{N=1}^{\infty} p^N T_N Z_0 \tag{F.7}$$

where

$$T_N Z_0 := \frac{1}{N} \sum_{s=1, s|N}^{N} \sum_{b=0}^{n-1} Z_0\Big(\frac{s\tau + b}{n}, y^s; \frac{s\overline{\tau} + b}{n}, \overline{y}^s\Big) \tag{F.8}$$

and $n = N/s$. Using (F.5) we see that this also holds for the case of the supertrace.

Appendix G. Examples of the index for iterated symmetric products

The partition function of the symmetric product theory $\mathrm{Sym}^N(\mathcal{S})$, with $N = Q_1 Q_5$, can be written in the compact form (4.5):

$$Z\Big(\mathrm{Sym}^N(\mathcal{S})\Big) = T_N Z_0 + \ldots \tag{G.1}$$

where T_N denotes the Hecke operator, and the dots stand for the higher-order terms which do not contribute to the index (see section 4). Similarly, the partition function of the iterated symmetric product theory looks like

$$Z\Big(\mathrm{Sym}^{Q_1}\ \mathrm{Sym}^{Q_5}(\mathcal{S})\Big) = T_{Q_1} T_{Q_5} Z_0 + \ldots \tag{G.2}$$

Our goal here is to compare the massless spectrum of these two theories. Using the properties of the Hecke operators summarized in section 4.2, we can write the linear term in (G.1) as

$$T_N Z_0 = \prod_p T_{p^{e_p}} Z_0 \tag{G.3}$$

where $N = Q_1 Q_5 = \prod p^{e_p}$. Similarly, factorizing $Q_1 = \prod p^{e'_p}$ and $Q_5 = \prod p^{e''_p}$, we have

$$T_{Q_1} T_{Q_5} Z_0 = \prod_p T_{p^{e'_p}} T_{p^{e''_p}} Z_0 \tag{G.4}$$

In particular, from (4.17) it follows that, when Q_1 and Q_5 are relatively prime, the partition function (G.3) of the symmetric product theory is equal to the partition function (G.4) of the iterated symmetric product theory. Hence, in order to see a difference between these two theories, one should consider Q_1 and Q_5 which are *not* relatively prime.

In order to see the difference between (G.3) and (G.4) when the prime factor p occurs in both Q_1 and Q_5, it is instructive to consider a simple case

$$Q_1 = p^{r_1} \quad , \quad Q_5 = p^{r_2} \tag{G.5}$$

so that

$$N = p^r \quad , \quad r = r_1 + r_2 \tag{G.6}$$

In this case, in order to evaluate the difference[9]

$$Z\Big(\mathrm{Sym}^{Q_1}\ \mathrm{Sym}^{Q_5}(\mathcal{S})\Big) - Z\Big(\mathrm{Sym}^{Q_1 Q_5}(\mathcal{S})\Big) = T_{p^{r_1}} T_{p^{r_2}} Z_0 - T_{p^r} Z_0 + \ldots \tag{G.7}$$

[9] Again, the dots stand for the higher order terms that do not contribute to the index.

we need to find a relation between the Hecke operators $T_{p^{r_1}} T_{p^{r_2}}$ and T_{p^r}. Using (4.18), we find the desired identity

$$T_{p^{r_1}} T_{p^{r_2}} = \sum_{k=0}^{\min(r_1,r_2)} \frac{1}{p^k} T_{p^{r_1+r_2-2k}} W_{p^k} \tag{G.8}$$

where we also used multiplicativity of W_p, $W_p^k = W_{p^k}$.

Now, substituting (G.8) into (G.7), we obtain

$$\begin{aligned}
&T_{p^{r_1}} T_{p^{r_2}} Z_0(\tau, z_\pm) - T_{p^r} Z_0(\tau, z_\pm) = \\
&\quad = \sum_{k=1}^{\min(r_1,r_2)} \frac{1}{p^k} T_{p^{r-2k}} W_{p^k} Z_0(\tau, z_\pm) = \sum_{k=1}^{\min(r_1,r_2)} \frac{1}{p^k} T_{p^{r-2k}} Z_0(\tau, z_\pm^{p^k}) = \\
&\quad = \sum_{k=1}^{\min(r_1,r_2)} \frac{1}{p^k} \frac{1}{p^{r-2k}} \sum_{ad=p^{r-2k}} \sum_{b=0}^{d-1} Z_0 \left(\frac{a\tau + b}{d}, z_\pm^{ap^k} \right) = \\
&= \sum_{k=1}^{\min(r_1,r_2)} \frac{1}{p^{r-k}} \sum_{\delta=0}^{r-2k} \sum_{b=0}^{p^\delta - 1} Z_0 \left(p^{r-2k-2\delta} \tau + \frac{b}{p^\delta}, z_\pm^{p^{r-k-\delta}} \right)
\end{aligned} \tag{G.9}$$

Therefore, the problem reduces to evaluating the terms of the form $T_{p^{r-2k}} Z_0(\tau, z_\pm^{p^k})$, which is similar to the problem studied in section 5.8. In particular, among the massless states of the iterated symmetric product theory $\mathrm{Sym}^{Q_1}\, \mathrm{Sym}^{Q_5}(\mathcal{S})$, which are not contained in the $\mathrm{Sym}^{Q_1 Q_5}(\mathcal{S})$ theory, we find the states of the form (5.87) where $a = p^{r-k-\delta}$:

$$\bigoplus_{\substack{(n_0,k,\delta,s) \\ \alpha+\beta=(a-1)}} \left(\frac{2an_0 + \alpha + 1}{2}, \frac{2an_0 + \beta + 1}{2}; \frac{am + \alpha + 1}{2}, \frac{am + \beta + 1}{2} \right) \tag{G.10}$$

As in (5.84) – (5.85), there are two families of solutions for n_0 and m:

$$\begin{aligned}
1) \qquad & m = 2n_0 + sp^\delta \qquad s = -1, 0, 1, \ldots, p^{r-2k-\delta} - 1 \\
& \max\left(0, -\frac{sp^\delta}{2}\right) \le n_0 \le \min\left(p^\delta - 1, \frac{(p^{r-2k-\delta} - s)p^\delta - 1}{2}\right)
\end{aligned} \tag{G.11}$$

and

$$\begin{aligned}
2) \qquad & m = sp^\delta - 2n_0 - 1 \qquad s = 1, 2, \ldots, p^{r-2k-\delta} + 1 \\
& \max\left(0, \frac{sp^\delta - p^{r-2k}}{2}\right) \le n_0 \le \min\left(p^\delta - 1, \frac{sp^\delta - 1}{2}\right)
\end{aligned} \tag{G.12}$$

In both cases, k runs from 1 to $\min(r_1, r_2)$ and $\delta = 0, \ldots, (r-2k)$, *cf.* (G.9). Note, that none of these extra states, which appear when g.c.d.$(Q_1, Q_5) > 1$, can be a supergravity state.

Example. $Q_1 = Q_5 = p$

Consider a simple example, where $r_1 = r_2 = 1$, so that $r = 2$. In this example, we have $k = 1$ and $\delta = 0$, so that only the term $Z_0(\tau, z_\pm^{p^2})$ appears on the right-hand side of (G.9). Moreover, the two families of solutions, (G.11) and (G.12), collapse to single solution

$$m = n_0 = 0 \tag{G.13}$$

Therefore, the spectrum (G.10) takes the following simple form:

$$\bigoplus_{\alpha+\beta=p-1} \left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}; \frac{\alpha+1}{2}, \frac{\beta+1}{2}\right)_R \tag{G.14}$$

After the spectral flow to the NS sector, we get

$$\bigoplus_{\beta=0}^{p-1} \left(\frac{p(p-1)-\beta}{2}\ ,\ \frac{\beta}{2}\ ;\ \frac{p(p-1)-\beta}{2}\ ,\ \frac{\beta}{2}\right)_{NS} \tag{G.15}$$

In addition, in general there are extra states (5.98) coming from non-trivial factorization of (5.93). The values of n_0 and m for these states are given by (5.96), where s_1 and s_2 assume their values in the range (5.97). As we discussed above, there are no additional states for $p = 2$. In this case, (G.10) – (G.12) is the complete answer.

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